Q3 Position of engine c.g. (given) $= \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix}$ m

Weight of engine (given) $= \begin{pmatrix} 0 \\ 0 \\ -50 \end{pmatrix}$ kN

a) Moment about origin $\underline{M} = \underline{r} \times \underline{F} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ -50 \end{pmatrix}$

$$\underline{M} = \begin{pmatrix} +50 \\ +250 \\ 0 \end{pmatrix} \text{ kNm}$$

b) line OT (given) $\Rightarrow \underline{t} = \begin{pmatrix} 30 \\ -14 \\ 2 \end{pmatrix}$ unit vector $\hat{\underline{t}} = \frac{1}{\sqrt{1100}} \begin{pmatrix} 30 \\ -14 \\ 2 \end{pmatrix}$

use scalar product to project $\underline{M}$ onto $\hat{\underline{t}}$

$$\underline{M}_t = (\underline{M} \cdot \hat{\underline{t}})\hat{\underline{t}} = \frac{1}{\sqrt{1100}} \underbrace{(50 \times 30 + (250 \times (-14)) + 0)}_{(1500 + (-3500)) = -2000} \underbrace{\left[\frac{1}{\sqrt{1100}} \begin{pmatrix} 30 \\ -14 \\ 2 \end{pmatrix}\right]}_{\hat{t}}$$

$$= -\frac{2000}{1100} \begin{pmatrix} 30 \\ -14 \\ 2 \end{pmatrix} = \begin{pmatrix} -54.5 \\ +25.5 \\ -3.6 \end{pmatrix} \text{ kNm}$$

This causes twisting of the wing about its axis.

c) $\underline{M}_{\perp} = \underline{M} - M_{parallel}$

$$\begin{pmatrix} 50 \\ 250 \\ 0 \end{pmatrix} - \begin{pmatrix} -54.5 \\ 25.5 \\ -3.6 \end{pmatrix} = \begin{pmatrix} 104.5 \\ 224.5 \\ +3.6 \end{pmatrix} \text{kNm}$$

check

$$\begin{pmatrix} 104.5 \\ 224.5 \\ 3.6 \end{pmatrix} \cdot \begin{pmatrix} 30 \\ -14 \\ 2 \end{pmatrix} = 0$$

(d) This component of the moment causes bending of the wing about its axis.

d) See answers to parts (b) and (c).

M 3

a) Equipollent force system from M2

$$\underline{F} = \begin{pmatrix} -2.4 \\ -2.7 \\ 0 \end{pmatrix} N \qquad \underline{M} = \begin{pmatrix} 0 \\ 0 \\ -1.89 \end{pmatrix} Nm$$

∴ Need a force and a moment to provide equilibrium

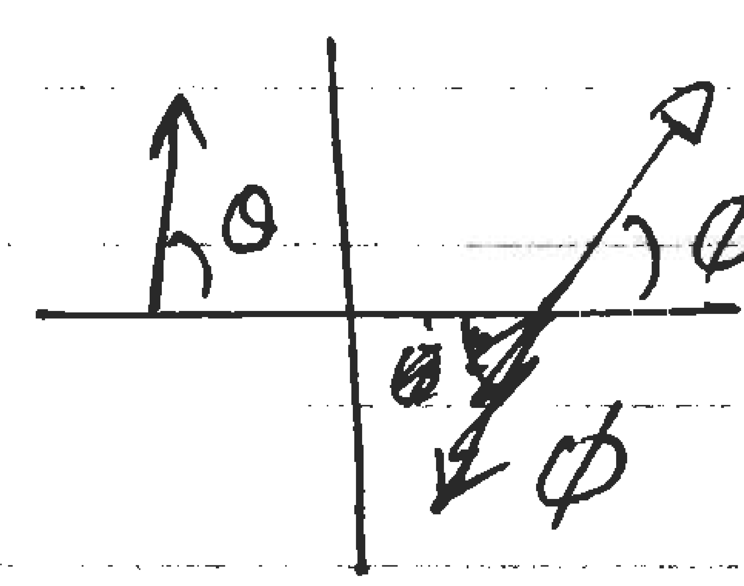
∴ a) No

b) No

c) Yes. $\underline{F} = \begin{pmatrix} +\cancel{2.4} \\ +\cancel{2.7} \\ 0 \end{pmatrix} N$, $\underline{M} = \begin{pmatrix} 0 \\ 0 \\ +1.89 \end{pmatrix} Nm$

d) Two forces, known magnitude, unknown direction
express in terms of angle to $x$ axis

$$\underline{F_1} = 20 \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$




$$\underline{F_2} = 20 \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}$$

$$\underline{F_1} + \underline{F_2} = \begin{pmatrix} +2.4 \\ +2.7 \\ 0 \end{pmatrix} = \begin{pmatrix} 20\cos\theta + 20\cos\phi \\ 20\sin\theta + 20\sin\phi \\ 0 \end{pmatrix} \begin{matrix} (1) \\ (2) \\ \\ \end{matrix}$$

$$\sum (\underline{r} \times \underline{F})_{\cancel{24}} = \begin{pmatrix} 0 \\ 0 \\ +1.89 \end{pmatrix}$$

$$20\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} = 20\begin{pmatrix} 0 \\ 0 \\ -\sin\theta \end{pmatrix}$$

$$20\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} = 20\begin{pmatrix} 0 \\ 0 \\ +\sin\phi \end{pmatrix}$$

$\therefore\ 20(-\sin\theta + \sin\phi) = +1.29$ (3)

add (2) + (3) $\Rightarrow$ $40\sin\phi = 4.59 \Rightarrow \sin\phi = 0.115$

substitute back into (2) $\Rightarrow$ $\sin\theta = \dfrac{-1.89}{20} + 0.115 = 0.020$

$\theta = 1.16°$, $\phi = 6.6°$ $\quad \cos\phi = 0.99$, $\cos\theta = 1.0$

or $\theta = 1.16$ $\quad \phi = 173.4$ $\quad \cos\phi = -0.99$ $\quad \cos\theta = 1.0$

~~$\therefore$ substitute back into (1)~~

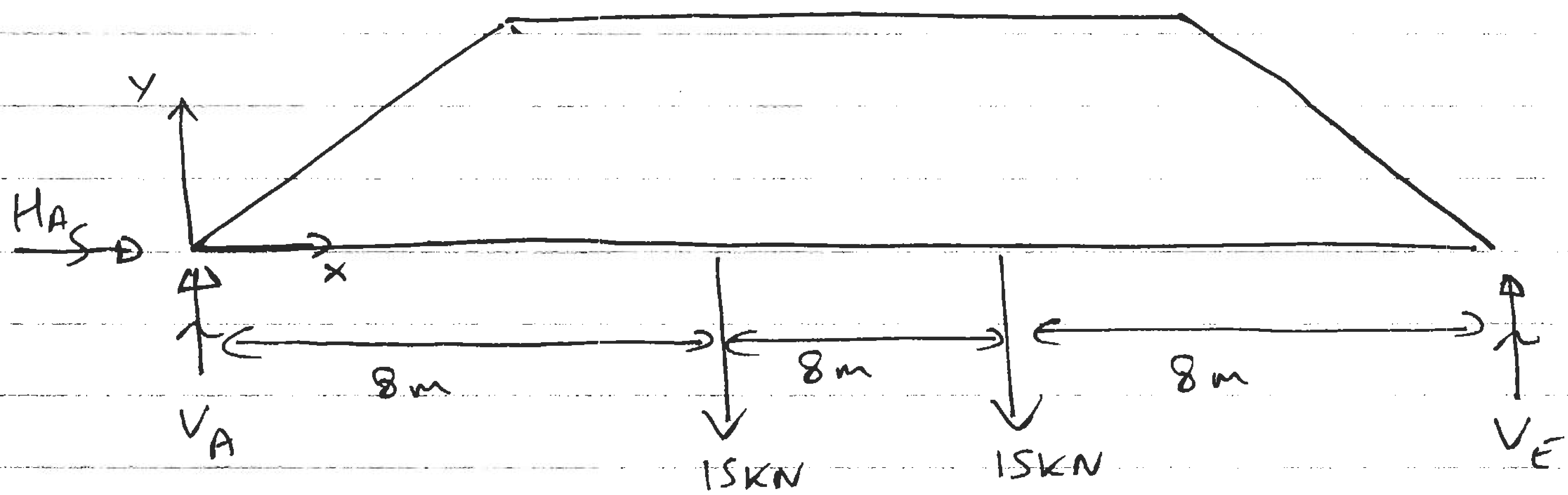
or $\theta = 178.8$ $\quad \phi = 1.16°$ $\quad \cos\phi = 0.99$ $\quad \cos\theta = -1.00$

$\therefore$ by inspection $\theta = 1.16°$, $\phi = 173.4° \Rightarrow$ (1) $= 0.2\,N$ $\quad \therefore$ not sufficient

Cannot ~~replace~~ achieve equilibrium with this pair of forces - problem is over constrained.

M4

a)




$$\sum \overrightarrow{F_x} = 0: \quad H_A + 0 = 0 \Rightarrow H_A = 0$$

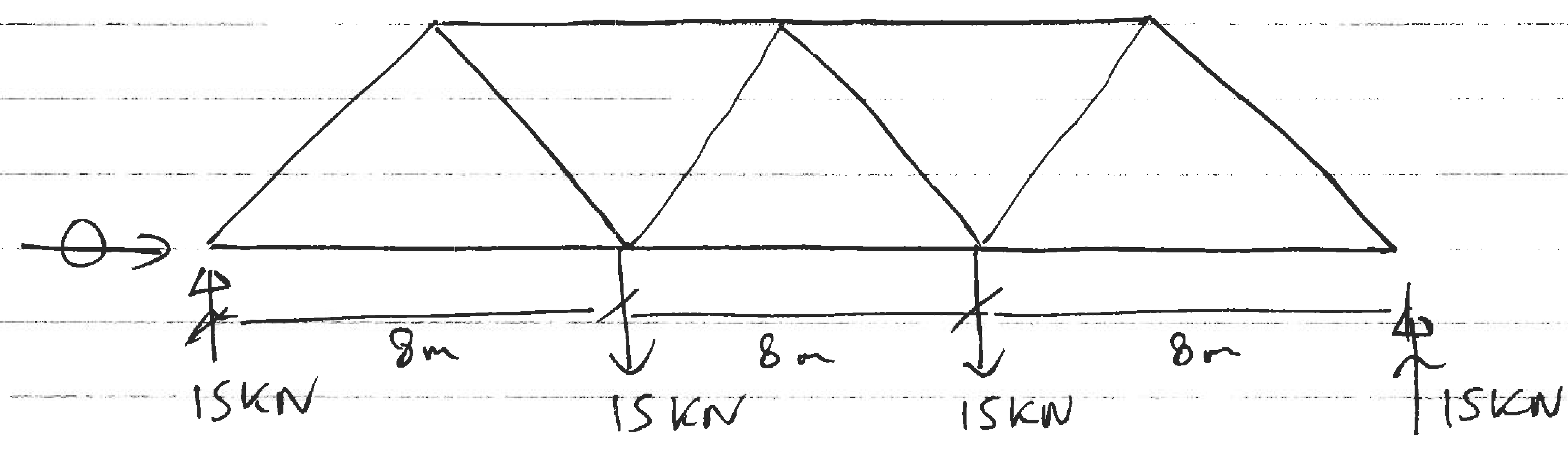
$$\sum F_y \uparrow = 0: \quad V_A + V_E - 30 = 0 \Rightarrow \quad V_A + V_E - 30 = 0$$

$$\sum M_A = 0: \quad -15 \times 8 - 15 \times 16 + 24 V_E = 0$$

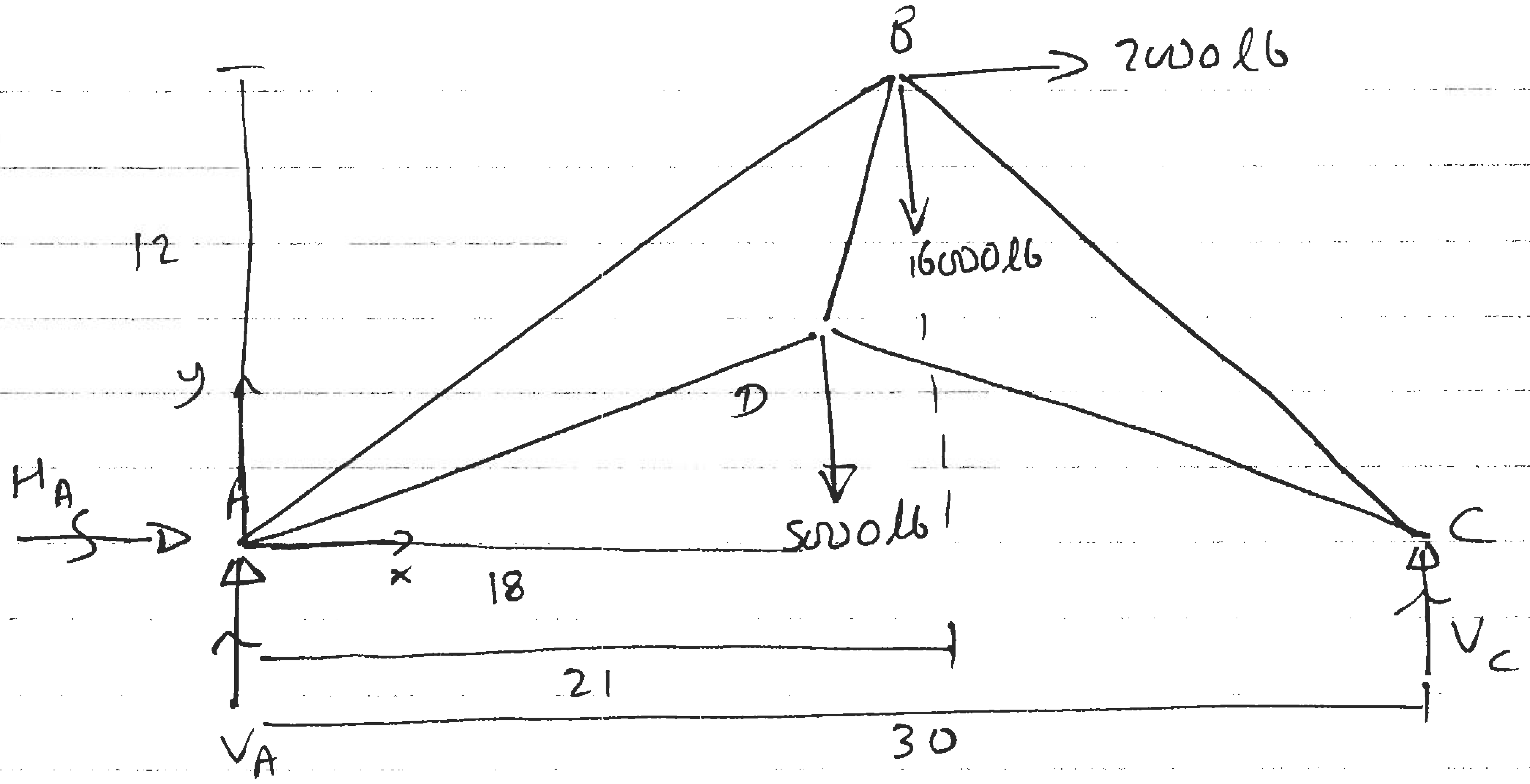
$$= V_E = 15\text{ KN}$$

$$\therefore V_A = 15\text{ KN}$$

must be equal by symmetry



6)




$\sum \vec{F}_x = 0$: $H_A + 2000 = 0 \Rightarrow H_A = -2000$ lb

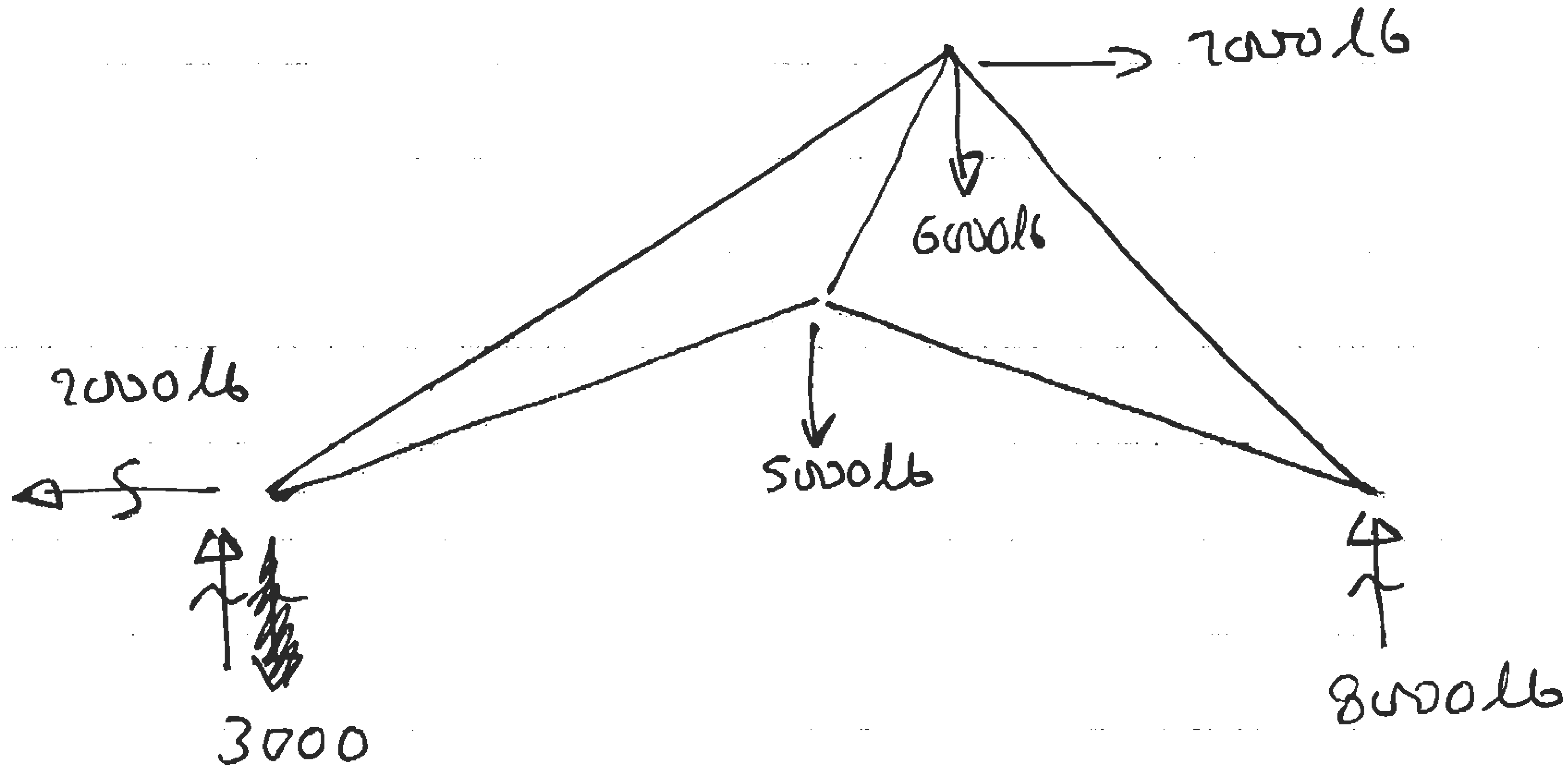
(i.e. in opposite direction to initial assumption)

$\sum F_y = 0$: $V_A - 5000 - 6000 + V_C = 0$

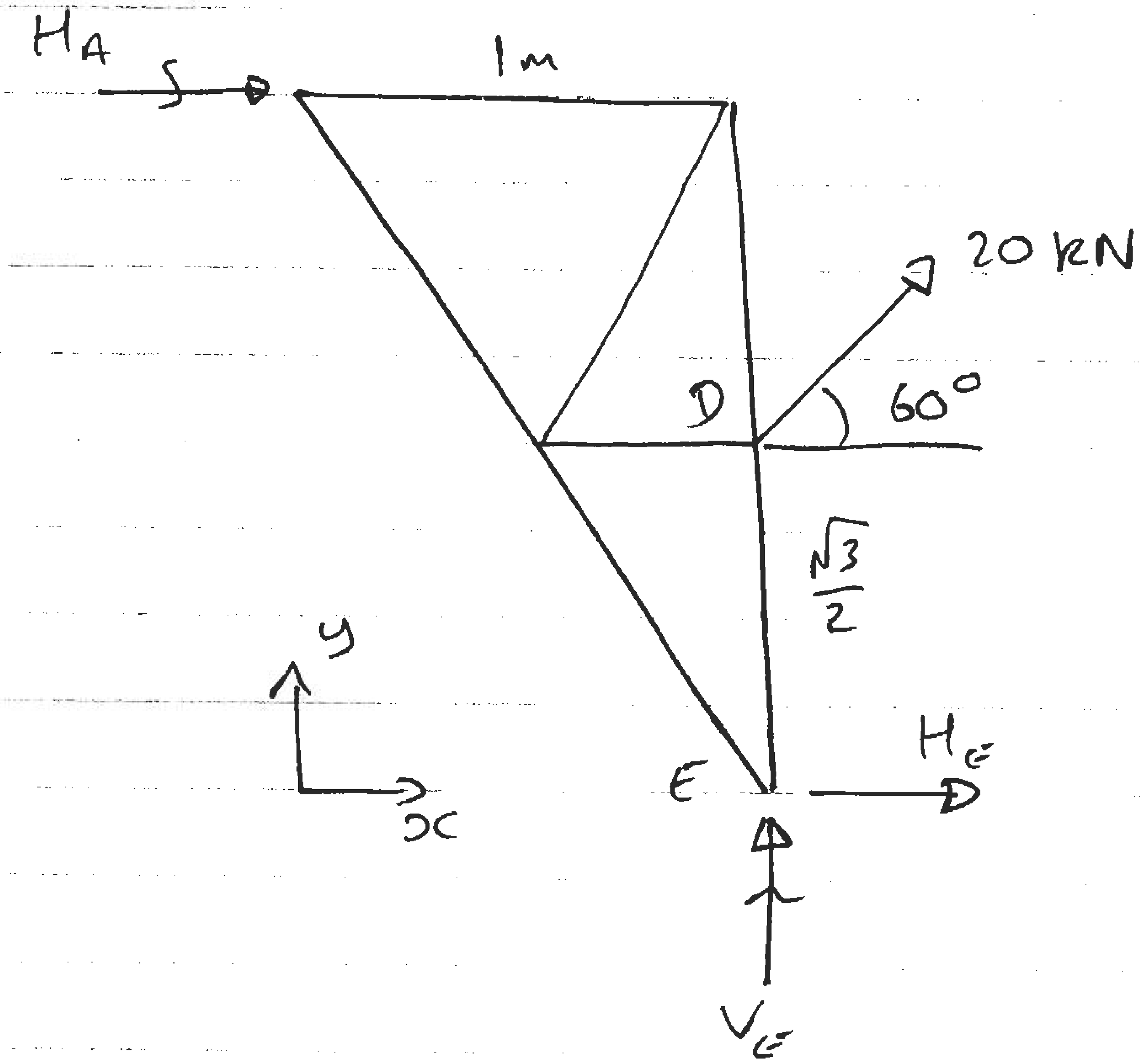
$\sum M_A = 0$: $-18 \times 5000 - 21 \times 6000 - 12 \times 2000 + V_C \times 30 = 0$

$\Rightarrow V_C = 8000$ lb

$\therefore V_A = +3000$ lb ~~i.e. in opposite direction~~



M4 c)



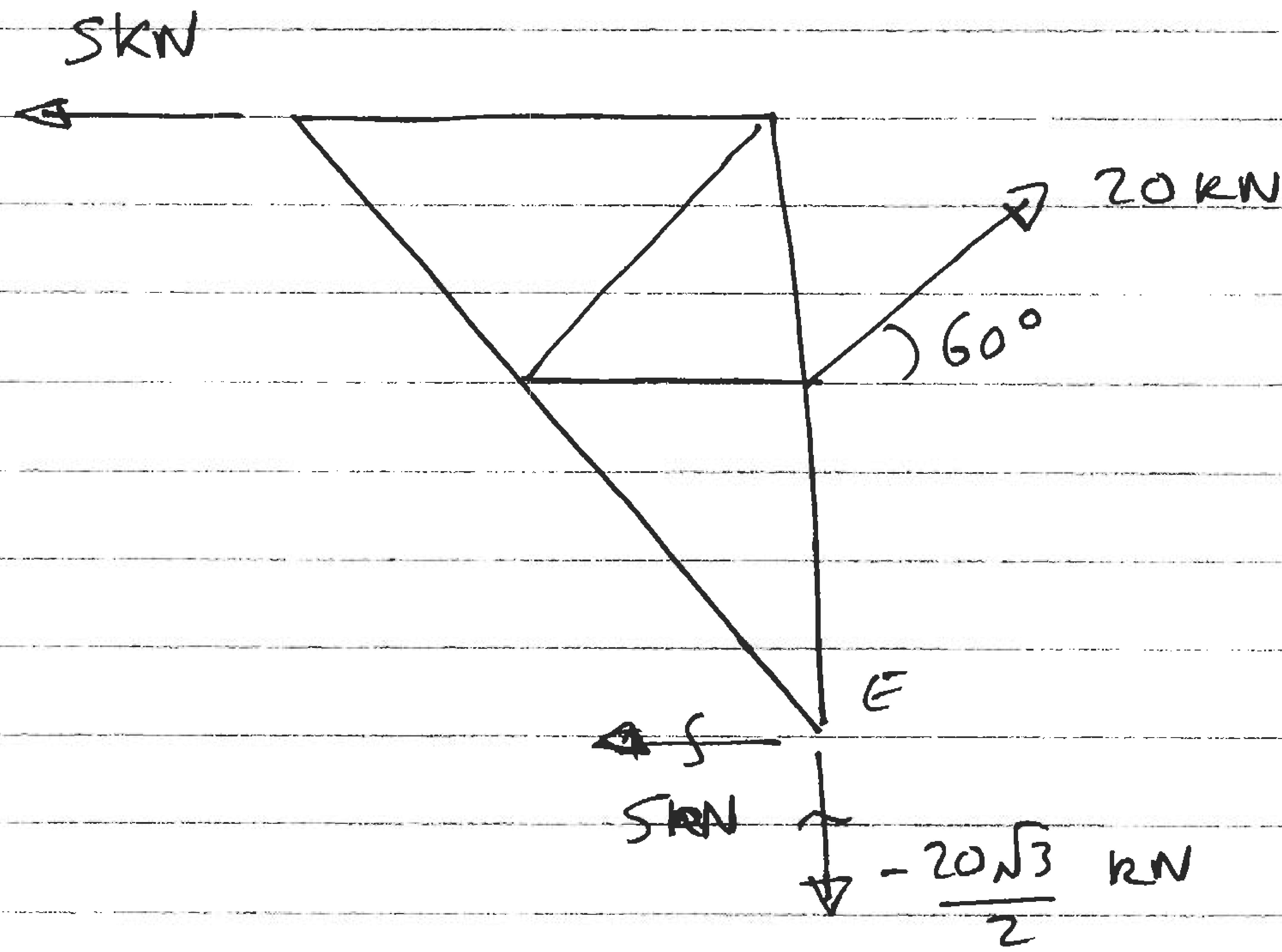

$$\Sigma \vec{F}_x = 0: \quad H_A + H_E + 20\cos 60^\circ = 0$$
$$H_A + H_E + 10 = 0$$

$$\Sigma F_y \uparrow = 0: \quad V_E + 20\sin 60^\circ = 0$$
$$V_E + 20\frac{\sqrt{3}}{2} = 0 \Rightarrow V_E = -20\frac{\sqrt{3}}{2} \text{ kN}$$

$$\Sigma \circlearrowright M_D = 0: \quad H_E\frac{\sqrt{3}}{2} - H_A\frac{\sqrt{3}}{2} = 0$$

eliminates force of 20kN, $V_E$

$$\therefore H_E = H_A \Rightarrow H_A = H_E = -5 \text{ kN}$$


5kN
20kN
60°
E
5kN
$-\frac{20\sqrt{3}}{2}$ kN

M2

Forces $2\begin{pmatrix}\frac{2}{\sqrt5}\\ -\frac{1}{\sqrt5}\\ 0\end{pmatrix}$ N @ $\begin{pmatrix}-3\\ 2\\ 0\end{pmatrix}$ $\quad 3\begin{pmatrix}-1\\ 0\\ 0\end{pmatrix}$ N @ $\begin{pmatrix}-1\\ -2\\ 0\end{pmatrix}$

$3\begin{pmatrix}-\frac{1}{\sqrt2}\\ -\frac{1}{\sqrt2}\\ 0\end{pmatrix}$ N @ $\begin{pmatrix}3\\ 3\\ 0\end{pmatrix}$

Moment $5\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$ Nm @ $\begin{pmatrix}2\\ -2\\ 0\end{pmatrix}$

a) $\sum \underline{F}$ : $2\begin{pmatrix}\frac{2}{\sqrt5}\\ -\frac{1}{\sqrt5}\\ 0\end{pmatrix} + 3\begin{pmatrix}-1\\ 0\\ 0\end{pmatrix} + 3\begin{pmatrix}-\frac{1}{\sqrt2}\\ -\frac{1}{\sqrt2}\\ 0\end{pmatrix} = \begin{pmatrix}\cancel{-2.4}\\ \cancel{-2.7}\\ 0\end{pmatrix}$ N

-3.3

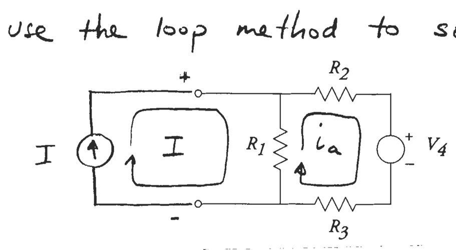
-3.01

b) $\sum \underline{r} \times \underline{F} + \sum \underline{M}$ : $2\begin{pmatrix}-3\\ 2\\ 0\end{pmatrix} \times \begin{pmatrix}\frac{2}{\sqrt5}\\ -\frac{1}{\sqrt5}\\ 0\end{pmatrix} = 2\begin{pmatrix}0\\ 0\\ -\frac{1}{\sqrt5}\end{pmatrix}$

$+ 3\begin{pmatrix}3\\ 3\\ 0\end{pmatrix} \times \begin{pmatrix}-\frac{1}{\sqrt2}\\ -\frac{1}{\sqrt2}\\ 0\end{pmatrix} = 3\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$

$+3\begin{pmatrix}-1\\ -2\\ 0\end{pmatrix} \times \begin{pmatrix}-1\\ 0\\ 0\end{pmatrix} = 3\begin{pmatrix}0\\ 0\\ -2\end{pmatrix} + \begin{pmatrix}0\\ 0\\ 5\end{pmatrix} \Rightarrow \sum \underline{M} + \sum \underline{r} \times \underline{F} = \begin{pmatrix}0\\ 0\\ -1\frac{2}{\sqrt5}\end{pmatrix}$ Nm

1. Add a test current to the terminals, and use the loop method to solve:




Apply KVL around the $i_a$ loop:

$$(R_1 + R_2 + R_3)\, i_a - I R_1 + V_4 = 0$$

$$\Rightarrow \quad (R_1 + R_2 + R_3)\, i_a = I R_1 - V_4$$

Plug in numbers:

$$(3 + 4 + 2)\, i_a = 3I - 12$$

$$\Rightarrow \quad i_a = \frac{1}{3} I - \frac{4}{3}$$

Now, the voltage across the terminals is

$$v = (I - i_a) R_1$$

$$= \left[ I - \left( {}^1\!/_3\, I - {}^4\!/_3 \right) \right] 3$$

$$= 2I + 4 = V_T + R_T I$$

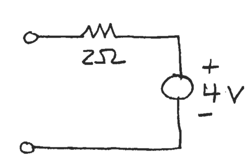
$$\Rightarrow \quad \boxed{R_T = 2\Omega, \quad V_T = 4\text{ V}}$$

Also, $I_N = V_T / R_T$, $R_N = R_T$. Therefore,

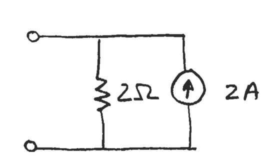
$$\boxed{I_N = 2A,\ R_N = 2\Omega}$$

Therefore, the equivalent circuits are:

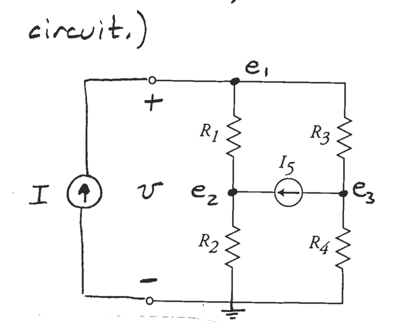
Thevenin:




Norton:




2. As above, apply a test current. This time, I'll use node method. (There are more unknown nodes with node method, but the loop method is trickier, due to current source in circuit.)




The node equations are

$e_1$: $(G_1 + G_3)\, e_1 \quad - G_1\, e_2 \quad - G_3\, e_3 = I$

$e_2$: $-G_2 e_1 + (G_1 + G_2)\, e_2 \quad + 0\, e_2 = I_5$

$e_3$: $-G_3 e_1 + \quad 0\, e_2 + (G_3 + G_4) e_3 = -I_5$

Plugging in numbers:

$$\begin{aligned} 1.25\,e_1 &- 1\,e_2 - 0.25\,e_3 = I \\ -1\,e_1 &+ 1.25\,e_2 + 0\,e_3 = 10 \\ -0.25\,e_1 &+ 0\,e_2 + 1.25\,e_3 = -10 \end{aligned}$$

Solve by Cramer's rule:

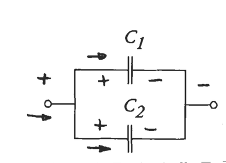
$$v = e_1 = \frac{\begin{vmatrix} I & -1 & -0.25 \\ 10 & 1.25 & 0 \\ -10 & 0 & 1.25 \end{vmatrix}}{\begin{vmatrix} 1.25 & -1 & -0.25 \\ -1 & 1.25 & 0 \\ -0.25 & 0 & 1.25 \end{vmatrix}}$$

$$= \frac{1.5625\,I + 9.375}{0.625}$$

$$= 15 + 2.5\,I$$

$$\Rightarrow \boxed{\begin{array}{c} V_T = 15\text{ V},\ R_T = R_N = 2.5\,\Omega \\ I_N = V_T / R_T = 6\text{ A} \end{array}}$$

1. (a)




The voltage across $C_1$ is the same as the voltage across the terminals, so

$$v_1 = v$$

Likewise,

$$v_2 = v$$

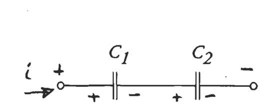
The total current into the + terminal is

$$\begin{aligned} i &= i_1 + i_2 \\ &= C_1 \frac{dv_1}{dt} + C_2 \frac{dv_2}{dt} \\ &= C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} \\ &= (C_1 + C_2) \frac{dv}{dt} \end{aligned}$$

Therefore, the equivalent capacitance is

$$\boxed{C = C_1 + C_2}$$

(b)




For the series connection,

$$i_1 = i_2 = i$$

Because the capacitors are in series,

$$v = v_1 + v_2$$

$$\Rightarrow \quad \dot{v} = \dot{v}_1 + \dot{v}_2$$

$$= \frac{i_1}{C_1} + \frac{i_2}{C_2}$$

$$= \frac{i}{C_1} + \frac{i}{C_2}$$

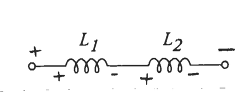
Therefore,

$$i = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1} v$$

$$= \frac{C_1 C_2}{C_1 + C_2} v$$

$$\Rightarrow \quad \boxed{C = \frac{C_1 C_2}{C_1 + C_2}}$$ is the equivalent capacitance

(c)




Because the inductors are in series,

$$i = i_1 = i_2$$

$$v = v_1 + v_2$$

$$= L_1 \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$

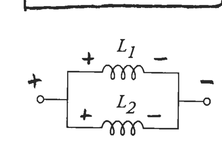
$$= L_1 \frac{di}{dt} + L_2 \frac{di}{dt}$$

$$= (L_1 + L_2) \frac{di}{dt}$$

Therefore, the equivalent inductance is

$$\boxed{L = L_1 + L_2}$$

(d)




Because the inductors are in parallel,

$$v = v_1 = v_2$$

$$i = i_1 + i_2$$

$$\Rightarrow \frac{di}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt}$$

$$= \frac{1}{L_1} v_1 + \frac{1}{L_2} v_2$$

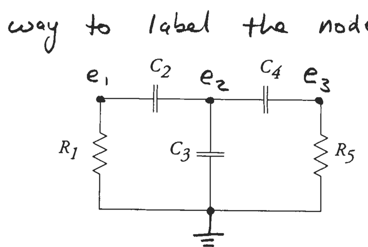
$$= \frac{1}{L_1} v + \frac{1}{L_2} v$$

$$= \left( \frac{1}{L_1} + \frac{1}{L_2} \right) v$$

Therefore, the equivalent inductance is

$$\boxed{L = \left( \frac{1}{L_1} + \frac{1}{L_2} \right)^{-1} = \frac{L_1 L_2}{L_1 + L_2}}$$

2. One way to label the nodes is:




Then the node equations are

$$e_1:\ \left(C_2\frac{d}{dt}+G_1\right)e_1 - C_2\frac{d}{dt}e_2 = 0$$

$$e_2:\ -C_2\frac{d}{dt}e_1 + \left(C_2\frac{d}{dt}+C_3\frac{d}{dt}+C_5\frac{d}{dt}\right)e_2 - C_4\frac{d}{dt}e_3 = 0$$

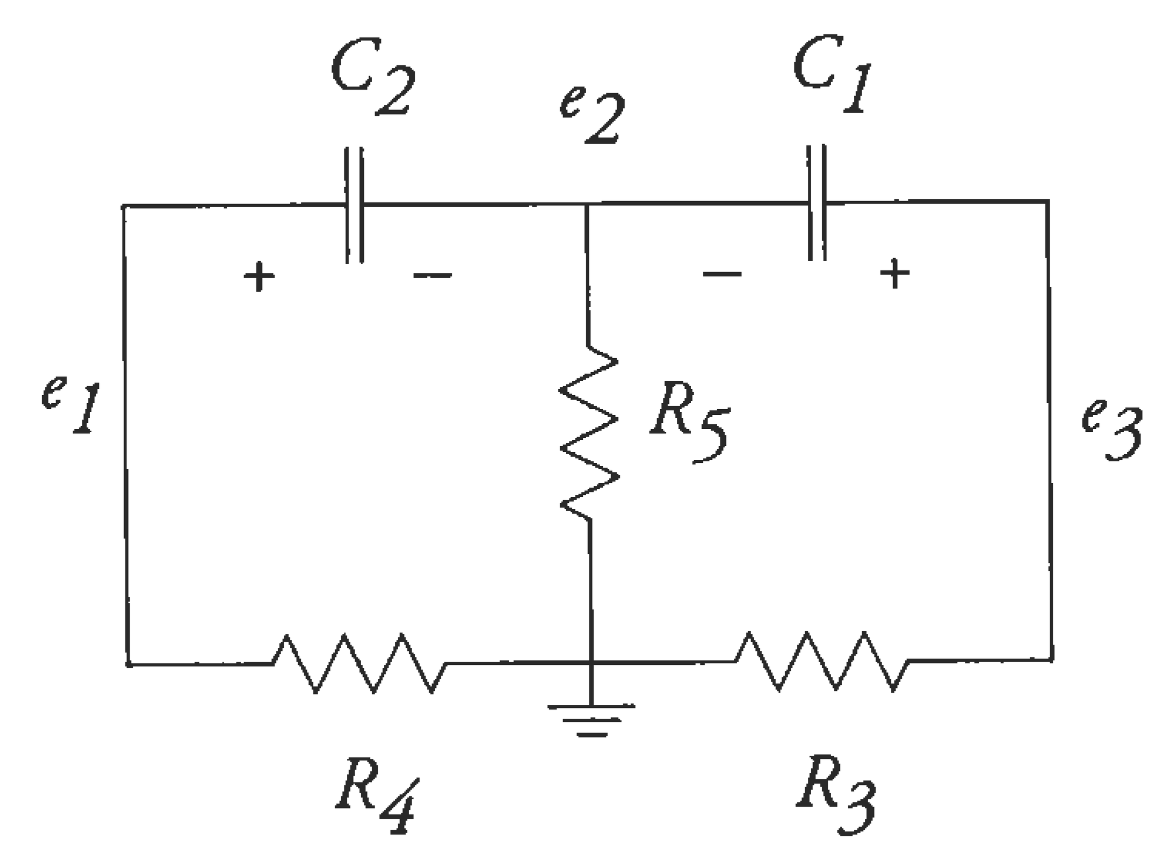
$$e_3:\ -C_4\frac{d}{dt}e_2 + \left(C_4\frac{d}{dt}+G_5\right)e_3 = 0$$

Plugging in component values,

$$\left(2\frac{d}{dt}+1\right)e_1 - 2\frac{d}{dt}e_2 = 0$$

$$-2\frac{d}{dt}e_1 + 8\frac{d}{dt}e_2 - 4\frac{d}{dt}e_3 = 0$$

$$-4\frac{d}{dt}e_2 + \left(4\frac{d}{dt}+0.2\right)e_3 = 0$$




The node equations are:

$e_1$: $$\left(C_2\frac{d}{dt} + G_4\right)e_1 - C_2\frac{d}{dt}e_2 = 0$$

$$-C_2\frac{d}{dt}e_1 + \left(C_1\frac{d}{dt} + C_2\frac{d}{dt} + G_5\right)e_2 - C_1\frac{d}{dt}e_3 = 0$$

$$-C_1\frac{d}{dt}e_2 + \left(C_1\frac{d}{dt} + G_3\right)e_3 = 0$$

Plugging in component values,

$$\left(2\frac{d}{dt} + 1\right)e_1 - 2\frac{d}{dt}e_2$$

$$-2\frac{d}{dt}e_1 + \left(3\frac{d}{dt} + 1\right)e_2 - \frac{d}{dt}e_3 = 0$$

$$-\frac{d}{dt}e_2 + \left(\frac{d}{dt} + 0.5\right)e_3 = 0$$

To find the solution, assume

$$e_1(t) = E_1 e^{st}$$
$$e_2(t) = E_2 e^{st}$$

Then

$$(2s+1)E_1 - 2sE_2 = 0$$

$$-2sE_1 + (3s+1)E_2 - sE_3 = 0$$

$$-sE_2 + (s+0.5)E_3 = 0$$

In matrix form,

$$\begin{bmatrix} 2s+1 & -2s & 0 \\ -2s & 3s+1 & -s \\ 0 & -s & s+0.5 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = 0$$

$$= M(s)\,\underline{E}$$

For there to be a nontrivial solution,

$$\det(M(s)) = 0$$
$$= 5s^2 + 3.5s + 0.5$$

This equation can be solved by using the quadratic formula, or a polynomial solver. The roots are

$$s_1 = -0.2\ \text{sec}^{-1}$$
$$s_2 = -0.5\ \text{sec}^{-1}$$

Solve for $\underline{E}$ in each case:

$\underline{s_1 = -0.2}$ $\Rightarrow M(s) = \begin{bmatrix} 0.6 & +0.4 & 0 \\ +0.4 & 0.4 & +0.2 \\ 0 & +0.2 & 0.3 \end{bmatrix}$ *

Normally, would solve by row reduction. Because of the zeros in M, can solve as follows: Set $E_3 = 1$. From last row of M,

$$+0.2E_2 + 0.3E_3 = 0$$ *

$$\Rightarrow E_2 = -1.5$$ *

From the circuit,

$$v_1(t) = e_3(t) - e_2(t)$$

$$v_2(t) = e_1(t) - e_2(t)$$

To match the initial conditions,

$$v_1(0) = 10\text{ V} = a(1+1.5)e^0 + b(1-0)e^0 \quad *$$

$$= 2.5a + b \quad *$$

$$v_2(0) = 0\text{ V} = a(1+1.5)e^0 + b(-0.5-0)e^0 \quad *$$

$$= 2.5a - 0.5b \quad *$$

Therefore,

$$\left.\begin{aligned} 2.5a + b &= 10 \\ 2.5a - 0.5b &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} a &= 1.333 \\ b &= 6.667 \end{aligned} \quad *$$

The final solution is then

$$\boxed{\begin{aligned} v_1(t) &= \left(+3.333\, e^{-0.2t} + 6.667\, e^{-0.5t}\right)\text{ V} \\ v_2(t) &= \left(3.333\, e^{-0.2t} - 3.333\, e^{-0.5t}\right)\text{ V} \end{aligned}}$$

N.B.: Corrected lines are marked with an asterisk.

From row 1 of M,

$$0.6\,E_1 + 0.4\,E_2 = 0$$

$$\Rightarrow E_1 = 1$$

*

So

$$\underline{E}_1 = \begin{bmatrix} 1 \\ -1.5 \\ 1 \end{bmatrix}$$

*

(Of course, any multiple of this is also a solution.)

$\underline{\underline{s_2 = -0.5}}$ $\Rightarrow M(s) = \begin{bmatrix} 0 & +1 & 0 \\ +1 & -0.5 & +0.5 \\ 0 & +0.5 & 0 \end{bmatrix}$ *

From row 1 (or row 3),

$$+E_2 = 0 \Rightarrow E_2 = 0$$ *

Arbitrarily choose $E_3 = 1$. Then from row 2,

$$+E_1 - 0.5\,E_2 + 0.5\,E_3 = 0$$ *

$$\Rightarrow E_1 = -0.5$$

Therefore,

$$\underline{E}_2 = \begin{bmatrix} -0.5 \\ 0 \\ 1 \end{bmatrix}$$ (or any multiple)

<u>Total solution</u>

The total solution is given by

$$\begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix} = a\,\underline{E}_1 e^{s_1 t} + b\,\underline{E}_2 e^{s_2 t}$$