Fluids – Lecture 6 Notes

1. Control Volumes
2. Mass Conservation
3. Control Volume Applications

Reading: Anderson 2.3, 2.4

Control Volumes

In developing the equations of aerodynamics we will invoke the firmly established and time-tested physical laws:
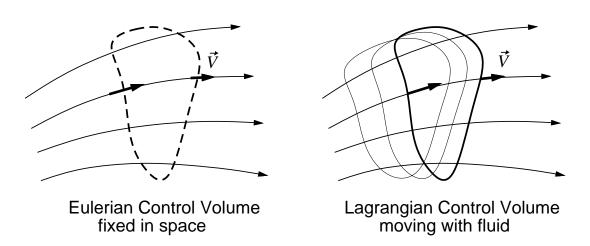
Conservation of Mass
Conservation of Momentum
Conservation of Energy

Because we are not dealing with isolated point masses, but rather a continuous deformable medium, we will require new conceptual and mathematical techniques to apply these laws correctly.

One concept is the *control volume*, which can be either finite or infinitesimal. Two types of control volumes can be employed:

1) Volume is fixed in space (Eulerian type). Fluid can freely pass through the volume's boundary.

2) Volume is attached to the fluid (Lagrangian type). Volume is freely carried along with the fluid, and no fluid passes through its boundary. This is essentially the same as the free-body concept employed in solid mechanics.

Eulerian Control Volume
fixed in space

Lagrangian Control Volume
moving with fluid

Both approaches are valid. Here we will focus on the fixed control volume.
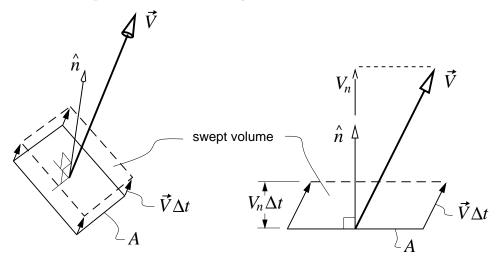
Mass Conservation

Mass flow

Consider a small patch of the surface of the fixed, permeable control volume. The patch has

area A, and normal unit vector $\hat{n}$. The plane of fluid particles which are on the surface at time t will move off the surface at time $t + \Delta t$, sweeping out a volume given by

$$\Delta \mathcal{V} = V_n \, A \, \Delta t$$

where $V_n = \vec{V} \cdot \hat{n}$ is the component of the veleocity vector normal to the area.

The mass of fluid in this swept volume, which evidently passed through the area during the Δt interval, is

$$\Delta m = \rho \, \Delta \mathcal{V} = \rho \, V_n \, A \, \Delta t$$

The *mass flow* is defined as the time rate of this mass passing though the area.

$$\text{mass flow} = \dot{m} = \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} = \rho \, V_n \, A$$

The *mass flux* is defined simply as mass flow per area.

$$\text{mass flux} = \frac{\dot{m}}{A} = \rho V_n$$

Mass Conservation Application
The conservation of mass principle can now be applied to the finite fixed control volume, but now it must allow for the possibility of mass flow across the volume boundary.

$$\frac{d}{dt} (\text{Mass in volume}) = \text{Mass flow into volume}$$

Using the previous relations we have

$$\frac{d}{dt} \iiint \rho \, d\mathcal{V} = - \oiint \rho \vec{V} \cdot \hat{n} \, dA \tag{1}$$

where the negative sign is necessary because $\hat{n}$ is defined to point outwards, so an inflow is where $-\vec{V} \cdot \hat{n}$ is positive.

Using Gauss's Theorem and bringing the time derivative inside the integral we have the result

$$\iiint \left[\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{V} \right) \right] d\mathcal{V} = 0$$

This relation must hold for *any* control volume whatsoever. If we place an infinitesimal control volume at every point in the flow and apply the above equation, we can see that the whole quantity in the brackets must be zero at every point. This results in the *Continuity Equation*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{V}\right) = 0 \tag{2}$$

which is the embodiment of the Mass Conservation principle for fluid flow. The steady flow version is:

$$\nabla \cdot \left(\rho \vec{V}\right) = 0 \qquad \text{(steady flow)}$$

For low-speed flow, steady or unsteady, the density ρ is essentially constant, which gives the very great simplification that the velocity vector field has zero divergence.

$$\nabla \cdot \vec{V} = 0 \qquad \text{(low-speed flow)}$$

Control Volume Applications

Steady Surface-Integral Form
All of the above forms of the continuity equation are used in practice. The surface-integral form (1) with the steady assumption,

$$\oiint \rho \vec{V} \cdot \hat{n} \, dA = 0 \tag{3}$$

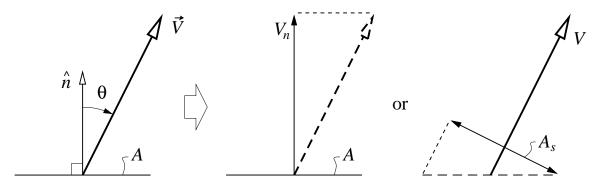is particularly useful in many engineering applications.

Use of equation (3) first requires construction of the fixed control volume. The boundaries are typically placed where the normal velocity and the density are known, so the surface integral can be evaluated piecewise. Placing the boundary along a solid surface is particularly useful, since here

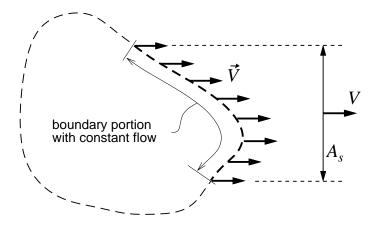$$\vec{V} \cdot \hat{n} = 0 \qquad \text{(at solid surface)}$$

and so that part of the boundary doesn't contribute to the integral. In the figure, we note that

$$\vec{V} \cdot \hat{n} \, dA = V \cos\theta \, dA = V_n \, dA = V \, dA_s$$

where dA_s is the area of the surface element projected onto a plane normal to $\vec{V}$.

The form of the integrand using dA_s is most useful on a portion of the boundary which has a constant value of $\rho \vec{V}$ (in magnitude and direction). The contribution of this boundary portion to the overall integral (3) is then

$$\oiint \rho \vec{V} \cdot \hat{n} \, dA = \rho V \oiint dA_s = \rho V A_s \qquad \text{(portion with constant } \rho \vec{V})$$

3

Note that the boundary portion does not need to be flat for A_s to be well defined. Only $\rho\vec{V}$ must be a constant.
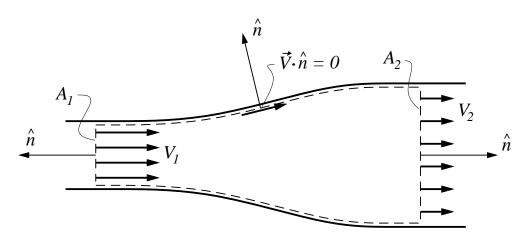
Channel Flow Application

Placing the control volume inside a pipe or channel of slowly-varying area, we now evaluate equation (3).

$$\oiint \rho\vec{V}\cdot\hat{n}\,dA \;=\; -\rho_1 V_1 A_1 \;+\; \rho_2 V_2 A_2 \;=\; 0$$
$$\rho_1 V_1 A_1 \;=\; \rho_2 V_2 A_2$$

The negative sign for station 1 is due to $\vec{V}\cdot\hat{n} = -V_1$ at that location. The control volume faces adjacent to walls do not contribute to the integral, since their normal vectors are perpendicular to the local flow and therefore have $\vec{V}\cdot\hat{n} = 0$.

Placing plane 2, say, at any other location, gives the general result that

$$\rho V A \;=\; \text{constant}$$

inside a channel. The product $\rho V A$ is also recognized as the constant channel mass flow.