Lecture S15

The Transfer Function

Often, we want to find response of system to sinusoidal or exponential inputs. Why?

- Exponential inputs produce exponential outputs
- Sine waves are also exponentials

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$= \text{Real}\left[e^{j\omega t} \right]$$

- Any input signal (almost) can be built up out of ~~sines~~ and ~~cosines~~, or exponentials. By superposition, can find output signal, if we can find response to exponentials.
- Sines and cosines are easily produced in the lab, and are frequently used to test systems.

So, what is response to an exponential input?

Assume our system is in state-space form, with

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$
$$y(t) = C\,x(t) + D\,u(t)$$

Suppose our input is

$$u(t) = V e^{st}$$

The solution will be

$$x(t) = \text{homogeneous solution} + \text{particular solution}$$

For now, we don't care about the homogeneous solution. What does particular solution look like? <u>Guess</u> that

$$x(t) = X e^{st}$$

Plug into differential equation and solve:

$$\dot{x}(t) = \frac{d}{dt} X e^{st}$$

$$= X s e^{st} = A X e^{st} + B U e^{st}$$

$$\Rightarrow \quad sX = AX + BU$$

$$\Rightarrow \quad (sI - A) X = BU$$

$$\Rightarrow \quad X = (sI - A)^{-1} B U$$

Given $x(t)$, $y(t)$ is

$$\begin{aligned} y(t) &= C x(t) + D u(t) \\ &= C X e^{st} + D U e^{st} \\ &= C (sI - A)^{-1} B U e^{st} + D U e^{st} \end{aligned}$$

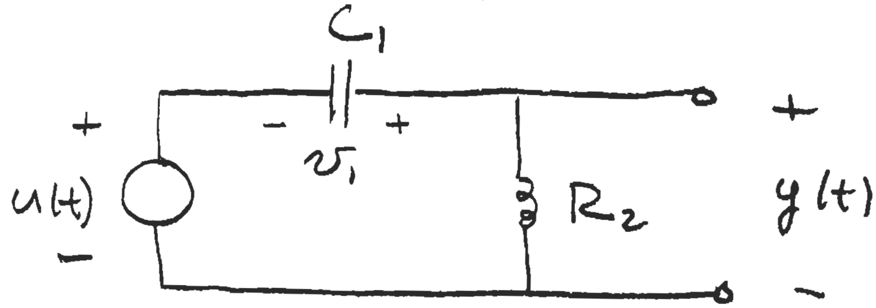
So

$$y(t) = Y e^{st}$$

where

$$Y = \underbrace{\left[C (sI - A)^{-1} B + D \right]}_{G(s) = \text{"transfer function"}} U$$

Example – "high-pass filter"



$C_1 = 1\,F$
$R_2 = 1\,\Omega$

Differential equation is

$$\dot{v}_1 = -\frac{1}{R_2C_1}\, v_1 - \frac{1}{R_2C_1}\, u$$

$$y = v_1 + u$$

$$A = \frac{-1}{R_2C_1} = -1 \qquad B = \frac{-1}{R_2C_1} = -1$$

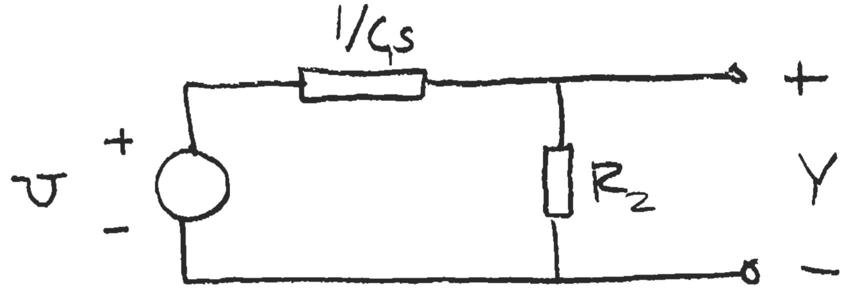
$$C = 1 \qquad D = 1$$

$$G(s) = C(sI-A)^{-1}B + D \qquad \left[= \frac{R_2C_1s}{R_2C_1s+1} \right]$$

$$= 1 \cdot (s+1)^{-1} \cdot (-1) + 1$$

$$= 1 - \frac{1}{s+1} = \frac{s}{s+1}$$

Note! Can also find $G(s)$ using impedances:



This is a voltage divider. So

$$Y = \frac{R_2 v}{R_2 + 1/C_1 s} = \underbrace{\frac{R_2 C_1 s}{R_2 C_1 s + 1}}_{G(s)} v$$

as before.

What is response of high pass filter to sinusoidal input?

$$u(t) = \cos \omega t = \text{Real}\Big[\underbrace{1}_{U} \cdot e^{\overset{s}{j\omega} t}\Big]$$

So take $U = 1$. Then

$$Y = G(j\omega)\, U$$

$$= \frac{j\omega}{j\omega + 1} \cdot 1 = \frac{j\omega}{j\omega+1} \; \frac{-j\omega+1}{-j\omega+1}$$

$$= \frac{\omega^2 + j\omega}{\omega^2+1}$$

$$\Rightarrow y(t) = \text{Real}\left[\left(\frac{\omega^2}{\omega^2+1} + j\frac{\omega}{\omega^2+1}\right) e^{j\omega t}\right]$$

$$\boxed{y(t) = \frac{\omega^2}{\omega^2+1} \cos \omega t - \frac{\omega}{\omega^2+1} \sin \omega t}$$

For large ω, $y(t) \approx \cos \omega t$

For small ω, $y(t) \approx -\frac{1}{\omega} \sin \omega t$

I.e, the filter "passes" high frequency sine waves, and attenuates low frequencies.