LECTURE S16

Analysis of Systems using Bilateral LT

To analyze systems using bilateral LT,

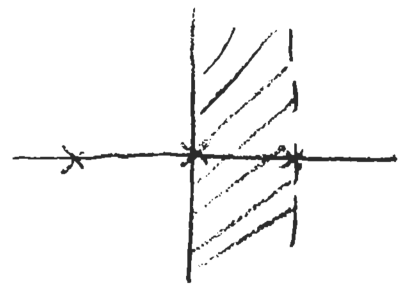
1) Keep track of region of convergence of each signal (impulse response, input)
2) If r.o.c. is not given explicitly, determine from conditions given
   - causality
   - direct LT of signal
3) r.o.c. of output is <u>intersection</u> r.o.c's of input, impulse response
4) Use r.o.c. to determine appropriate inverse LT.

Example What is the step response of the smoother?

Solution: $$G(s) = \frac{-a^2}{(s-a)(s+a)} \qquad -a < s < a$$

$$U(s) = \frac{1}{s} \qquad 0 < s$$

$$Y(s) = \frac{-a^2}{s(s-a)(s+a)} \qquad 0 < s < a$$

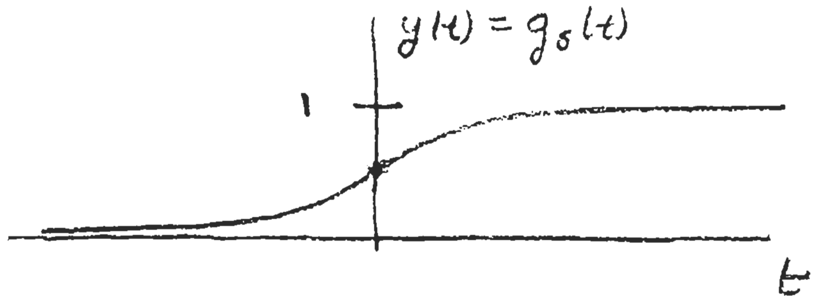


Partial Fractions:

$$Y(s) = \frac{1}{s} + \frac{-1/2}{s-a} + \frac{-1/2}{s+a}$$

$$(s > 0) \qquad (s < a) \qquad (s > -a)$$

$$\Rightarrow \quad y(t) = \begin{cases} 1 - \frac{1}{2} e^{-at}, & t \geq 0 \\ +\frac{1}{2} e^{at}, & t < 0 \end{cases}$$



This *is* a smoothed step!

BIBO Stability and the Bilateral LT

The conditions for stability derived for the unilateral LT apply to the bilateral LT as well.

<u>Fact</u> The (possibly noncausal) system G is BIBO stable if and only if

$$\int_{-\infty}^{\infty} |g(t)|\, dt = M < \infty$$

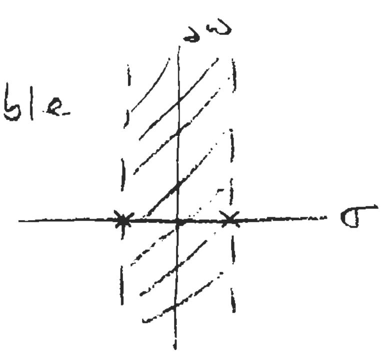
<u>Proof</u> same as for the unilateral LT.

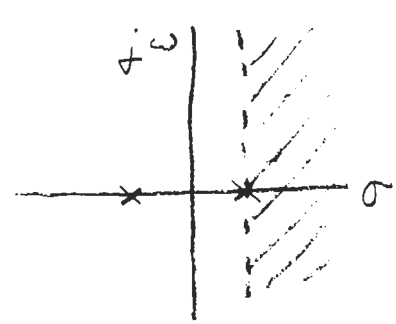
<u>Fact</u> The (possibly noncausal) system G is BIBO stable if and only if the LT of $g(t)$ converges over the region

$$\sigma_1 < \mathrm{Re}[s] < \sigma_2$$

where $\sigma_1 < 0$, $\sigma_2 > 0$. That is, the r.o.c. must contain the $j\omega$ axis.

stable





Proof: $G(s) = \int_{-\infty}^{\infty} g(t) e^{-st} dt$

$G(s)$ converges for $\mathrm{Re}[s] = 0$ $(s = j\omega)$

$$\iff \int_{-\infty}^{\infty} \left| g(t) e^{-j\omega t} \right| dt < \infty$$

$$= \int_{-\infty}^{\infty} |g(t)| \underbrace{\left| e^{-j\omega t} \right|}_{1} dt$$

$$= \int_{-\infty}^{\infty} |g(t)| dt$$

$\iff$ G is BIBO stable $\blacksquare$

So $j\omega$ axis is very important:

- stability depends on behavior of transform on $j\omega$ axis.
- For <u>causal</u> systems, system is stable if and only if there are no poles on or to the right of $j\omega$ axis.
- "Frequency response" (response to sinusoidal inputs) corresponds to transfer function for $s = j\omega$, and completely characterizes stable systems.

The Fourier Transform

Q: What happens if we restrict our consideration to stable signals?

A: We know that the LT converges for $\mathrm{Re}[s] = 0 \implies s = j\omega$.

So we can write $G(j\omega)$ instead of $G(s)$.

Fact: Knowing $G(s)$ for $s = j\omega$ is enough to determine $G(s)$ for all s in the r.o.c. This is called "analytic continuation."

So restricting s to $s = j\omega$ causes no loss of information.

The Fourier Transform (FT) is defined as

$$\boxed{\mathcal{F}[g(t)] = G(j\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt}$$

when the integral converges.

Note:

- It is conventional to write $G(j\omega)$, not $G(\omega)$. This reminds us that the FT is really a bilateral LT.
- There is a slightly different definition, which we will see shortly.

The Inverse Fourier Transform

Using LT results, we know that

$$g(t) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} G(j\omega) e^{st} \, ds$$

Change variables:

$$s = j\omega \implies ds = j\,d\omega$$
$$s = -j\infty \implies \omega = -\infty$$
$$s = +j\infty \implies \omega = +\infty$$

Therefore,

$$\boxed{g(t) = \mathcal{F}^{-1}[G(j\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega) e^{j\omega t} \, d\omega}$$

Written in this way, we can make some interesting observations:

1) Any (stable) signal can be expressed as a sum of sinusoids. Hence, the importance of the frequency response.

2) $\mathcal{F}^{-1}$ looks (almost) just like $\mathcal{F}$. The properties of $\mathcal{F}$ are similar to the properties of $\mathcal{F}^{-1}$!

3) $\mathcal{F}^{-1}$ is an integral along the real line $\Longrightarrow$ complex variable theory not needed, <u>in</u> <u>theory</u>. In practice, it's still very helpful.