LECTURE S17

The Fourier Transform

From last time:

$$\mathcal{F}: \quad G(j\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt$$

$$\mathcal{F}^{-1}: \quad g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega)\, e^{+j\omega t}\, d\omega$$

Note the pesky value of 2π that spoils the symmetry. Siebert uses:

$$\mathcal{F}: \quad G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j2\pi f t}\, dt$$

$$\mathcal{F}^{-1}: \quad g(t) = \int_{-\infty}^{\infty} G(f)\, e^{+j2\pi f t}\, df$$

which is more symmetric, but has pesky 2π in exponent.

f = frequency (Hz)

ω = circular frequency (rad/sec)

Properties of the FT

Connection to Laplace transform:

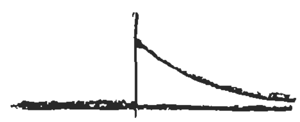
If $g(t)$ a stable signal,

$$G(j\omega) = G(s)\Big|_{s=j\omega}$$

$$\left[G(f) = G(s)\Big|_{s=j2\pi f} \right]$$

Ex|

$$g(t) = e^{-at}\sigma(t)$$



$$G(j\omega) = \frac{1}{j\omega + a}$$

$$G(f) = \frac{1}{j2\pi f + a} \qquad (\text{ick!})$$

Conjugate symmetry:

If $g(t)$ real [which is usually true],

$$G(-j\omega) = G^*(j\omega)$$

This follows from the fact that

$$e^{-j\omega t} = \left(e^{j\omega t}\right)^*$$

and the FT integral

<u>Sine/Cosine representation of signal:</u>
The inverse FT can be written as

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)\, e^{j\omega t}\, dt$$

$$= \frac{1}{2\pi}\left[\int_{0}^{\infty} G(j\omega)\, e^{j\omega t}\, dt + \underbrace{\int_{-\infty}^{0} G(j\omega)\, e^{j\omega t}\, dt}_{\int_0^\infty G(-j\omega)\, e^{-j\omega t}\, dt}\right]$$

$$= \frac{1}{2\pi}\int_{0}^{\infty}\left[G(j\omega)\, e^{j\omega t} + \left(G(j\omega)\, e^{j\omega t}\right)^*\right] dt$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \mathrm{Re}\left[G(j\omega)\, e^{j\omega t}\right] dt$$

$$G(j\omega) = \underbrace{G_R(j\omega)}_{\text{real part}} + \underbrace{G_I(j\omega)}_{\text{imag. part}}$$

also, $e^{j\omega t} = \cos\omega t + j\sin\omega t$

$\Rightarrow$

$$g(t) = \frac{1}{\pi}\int_0^\infty \left\{ G_R(j\omega)\cos\omega t - G_I(j\omega)\sin\omega t \right\} d\omega$$

That is, every stable signal $g(t)$ can be expressed as a sum of sines and cosines.

<u>Time-Frequency Duality</u>

Because of the symmetry of the FT and IFT, there is a <u>duality</u> between time and frequency:

If $\mathcal{F}[g(t)] = G(f)$, then

$$\mathcal{F}[G(t)] = g(-f)$$

$$\left(\begin{aligned} \mathcal{F}[g(t)] &= G(j\omega) \equiv f(\omega) \\ \mathcal{F}[f(\omega)] &= 2\pi\, g(-\omega) \end{aligned} \right)$$

Note factor of 2π

<u>Example</u> What is FT of unit step?

[Do Concept Test here]

<u>Answer</u> There is no FT! The unit step, $g(t) = \sigma(t)$, is <u>not</u> a stable signal, but we would like there to be a FT.

There is, in a limiting sense.

Before we do that one, let's do a different problem, as a warmup.

<u>Problem</u> What is $\mathcal{F}[g(t)]$, where

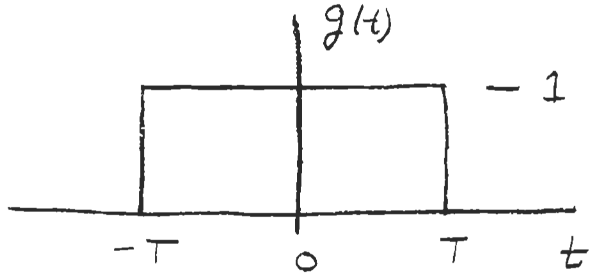
$$g(t) = 1, \quad \text{all } t\,?$$

Clearly, $g(t)$ is not stable. Instead, use

$$g(t) = \begin{cases} 1, & |t| \le T \\ 0, & |t| > T \end{cases}$$

and take limit as $T \to \infty$.

Example $g(t) = \begin{cases} 1, & |t| \le T \\ 0, & |t| > T \end{cases}$



What is $G(j\omega)$?

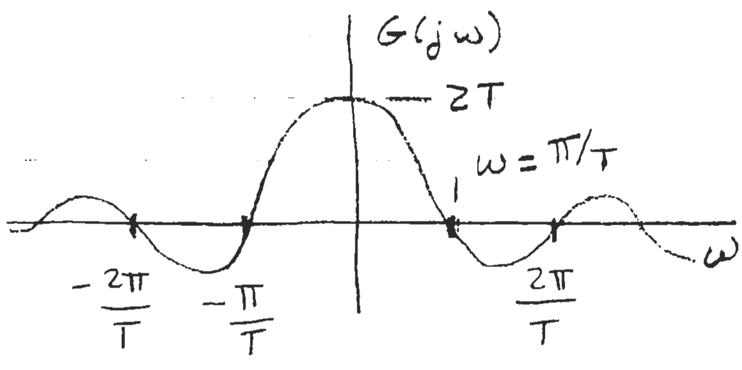
$$G(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} g(t)\, dt$$

$$= \int_{-T}^{T} e^{-j\omega t} \cdot 1\, dt$$

$$= \frac{1}{-j\omega} e^{-j\omega t} \Big|_{t=-T}^{T}$$

$$= \frac{e^{-j\omega T} - e^{+j\omega T}}{-j\omega} = 2\,\frac{\sin \omega T}{\omega} = 2T\,\frac{\sin \omega T}{\omega T}$$

$$= 2T \operatorname{sinc} \omega T$$



[Note that $G(j\omega)$ is real, because $g(t)$ is symmetric ($g(-t) = g(t)$).]

Note that height is proportional to T; width is proportional to $1/T$.

$\Rightarrow$ like an impulse for large T!

What is the area of the impulse?

$$A = \int_{-\infty}^{\infty} 2T \frac{\sin \omega T}{\omega T} \, d\omega = ???$$

This won't work. Instead, use

$$A = \int_{-\infty}^{\infty} G(j\omega) \, d\omega$$

$$= \left. \int_{-\infty}^{\infty} G(j\omega) e^{+j\omega t} \, d\omega \right|_{t=0}$$

$$= 2\pi \, g(0) = 2\pi$$