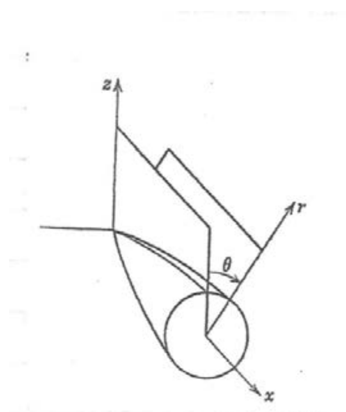
Bodies of Revolution: Slender Body Theory

1 COORDINATES

We will use cylindrical cordinates, x, r, θ

2 VELOCITY COMPONENTS (x, r, θ)



$$u_1 = U_\infty + u = \frac{\partial \Phi}{\partial x}$$

$$v = \frac{\partial \Phi}{\partial r}$$

$$\omega = \frac{1}{r}\frac{\partial \Phi}{\partial \theta}$$

3 Continuity equation

$$\frac{\partial}{\partial x}(\rho u_1) + \frac{1}{r}\frac{\partial}{\partial ur} + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho\omega) = 0$$

4 Linearized perturbation potential equation

$$(1 - M_\infty^2)\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0$$

5 Boundary conditions

Velocity gradients near axis are very large. Continuity equation yields:

$$\frac{1}{r}\frac{\partial}{\partial r}(vr) \sim \frac{\partial u}{\partial x}$$

or

$$\frac{\partial}{\partial r}(vr) \sim r\frac{\partial u}{\partial x}$$

Where $\frac{\partial u}{\partial x}$ is NOT infinite. Hence, for $r \to 0$, near the axis:

$$\frac{\partial}{\partial r}(vr) \sim 0$$

$$vr = a_0(x)$$

We may use a power series expansion:

$$vr = a_0 + a_1 r + a_2 r^2 + \ldots$$

or

$$vr = \frac{a_0}{r} + a_1 + a_2 r + \ldots$$

The correct statement of the boundary condition on the axis is:

$$\frac{dR}{dx} = \left(\frac{v}{U_\infty + u}\right)_R$$

Multiply by R:

$$R\frac{dR}{dx} = R\left(\frac{v}{U_\infty + u}\right) \cong \frac{(vr)_0}{U_\infty}$$

For irrotational flow:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}$$

Substituting for u:

$$\frac{\partial u}{\partial r} = \frac{a_0'}{r} + a_1' + a_2' r + \ldots$$

$$a_n' = \frac{\partial a_n}{\partial x}$$

Integrating:

$$u = a_0' log(r) + a_1' r + \ldots$$

And the "linearized" pressure coefficient reduces to:

$$C_p = -\frac{2u}{U_\infty} - \left(\frac{v}{U_\infty}\right)^2$$

$$C_p = \frac{2}{\gamma M_\infty^2}\left[\left[1 - \frac{\gamma - 1}{2} M_\infty^2 \left(\frac{2u}{U_\infty} + \frac{u^2}{U_\infty^2} + \frac{v^2}{U_\infty^2} + \frac{\omega^2}{U_\infty^2}\right)\right]^{\frac{\gamma}{\gamma-1}} - 1\right]$$

Axially symmetric flow:

- No variation with θ
- Conditions are the same in every meridian plane
- $\omega = 0$

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + (1 - M_\infty^2)\frac{\partial^2 \Phi}{\partial x^2} = 0$$

6 Incompressible solution

$$M_\infty = 0$$

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial x^2} = 0$$

This is Laplace's Equation and has the basic solution:

$$\Phi = \frac{Constant}{\sqrt{x^2 + r^2}}$$

This is a source of finite strength. For a source at the position $x = \xi$ on the x-axis, we write:

$$\Phi(x, r) = -\frac{A}{\sqrt{(x-\xi)^2 + r^2}}$$

Superposition is correct!

$$\Phi(x, r) = -\left[\frac{A_0}{\sqrt{x^2 + r^2}} + \frac{A_1}{\sqrt{(x-\xi_1)^2 + r^2}} + \frac{A_2}{\sqrt{(x-\xi_2)^2 + r^2}} + \ldots\right]$$

For a source distribution, we have:

$$\Phi(x, r) = -\int_0^l \frac{f(\xi)}{\sqrt{(x-\xi)^2 + r^2)}} d\xi$$

How is $f(\xi)$ determined?

7 Subsonic solution

Let: $m^2 \equiv (1 - M_\infty^2) > 0$

$$\frac{1}{m^2}\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{m^2}\frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{\partial^2\Phi}{\partial x^2} = 0$$

Transform as follows:

$$r' = mr$$

Substitute:

$$\frac{\partial^2\Phi}{\partial r'^2} + \frac{1}{r'}\frac{\partial\Phi}{\partial r'} + \frac{\partial^2\Phi}{\partial x^2} = 0$$

Solutions:

$$\Phi(x, r) = -\frac{A}{\sqrt{(x-\xi)^2 + m^2 r^2)}}$$

$$\Phi(x, r) = -\int_0^l \frac{f(\xi)}{\sqrt{(x-\xi)^2 + m^2 r^2)}} d\xi$$

8 F(x), S(x), R(x) Relation

For a line source, we let A be the volume of fluid sent out per unit time per unit length of the line source. Recall the line source:

$$\Phi(x,r) = \frac{A}{\sqrt{(x^2+r^2)}}$$

For a distribution of sources, we write:

$$\Phi(x,r) = \frac{A(x)}{\sqrt{(x^2+r^2)}} = \frac{f(x)}{\sqrt{(x^2+r^2)}}$$

At a distance r, the flow is distributed uniformly over a cylindrical surface with a circumference of $2\pi r$. Hence, at any x:

$$v = \frac{f(x)}{2\pi r} = \frac{\partial\Phi}{\partial r}$$

At the surface of the slender body:

$$\left(\frac{v}{U_\infty + u}\right)_R = \frac{dR}{dx}$$

or

$$\left(\frac{v}{U_\infty}\right)_R = \frac{dR}{dx}$$

Therefore:

$$v_R = U_\infty \frac{dR}{dx}$$

Substituting:

$$\frac{f(x)}{2\pi R} = U_\infty \frac{dR}{dx}$$

$$f(x) = 2\pi U_\infty R \frac{dR}{dx}$$

Any any x:

$$S(x) = \pi R^2$$

Therefore:

$$S'(x) = 2\pi R \frac{dR}{dx}$$

Hence:

$$f(x) = U_\infty S'(x)$$