Unit 10
St. Venant Torsion Theory

Readings:

Rivello	8.1, 8.2, 8.4
T & G	101, 104, 105, 106

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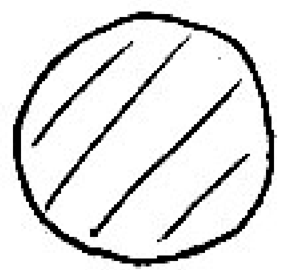
III. Torsion

We have looked at basic in-plane loading. Let's now consider a second "building block" of types of loading: *basic* *torsion.*

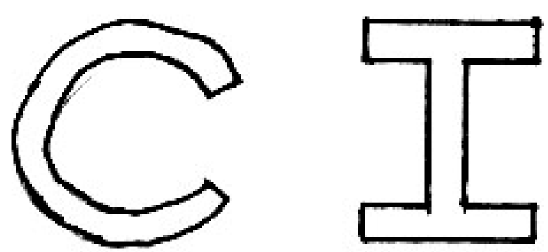
There are 3 basic types of behavior depending on the type of cross-section:

1. Solid cross-sections



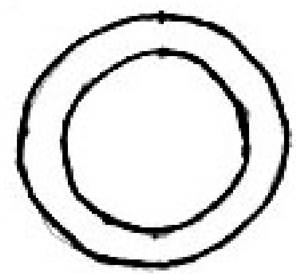
"classical" solution technique via stress functions

2. Open, thin-walled sections



Membrane Analogy

3. Closed, thin-walled sections



Bredt's Formula

In Unified you developed the basic equations based on some broad assumptions. Let's…

- Be a bit more rigorous
- Explore the limitations for the various approaches
- Better understand how a structure "resists" torsion and the resulting deformation
- Learn how to model general structures by these three basic approaches

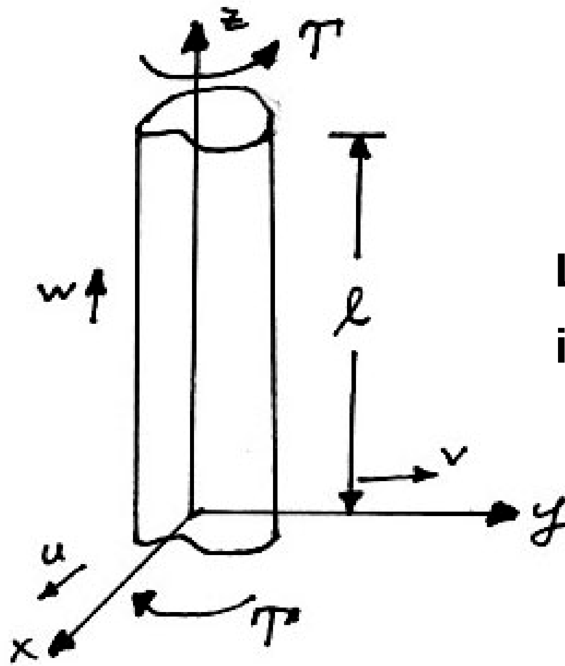
Look first at

Classical (St. Venant's) Torsion Theory

Consider a long prismatic rod twisted by end torques:

T [in - lbs] [m - n]

Figure 10.1 **Representation of general long prismatic rod**



Length (l) >> dimensions in x and y directions

Do not consider *how* end torque is applied (St. Venant's principle)

Assume the following **geometrical** **behavior:**

a) Each cross-section (@ each z) rotates as a rigid body (No "distortion" of cross-section shape in x, y)

b) Rate of twist, k = constant

c) Cross-sections are free to warp in the z-direction but the warping is the same for all cross-sections

This is the "St. Venant Hypothesis"

"warping" = extensional deformation in the direction of the axis about which the torque is applied

Given these assumptions, we see if we can satisfy the equations of elasticity and B.C.'s.

$\Rightarrow$ SEMI-INVERSE METHOD

Consider the deflections:

Assumptions imply that at any cross-section location z:

$$\alpha = \left(\frac{d\alpha}{dz}\right) z = k\, z$$

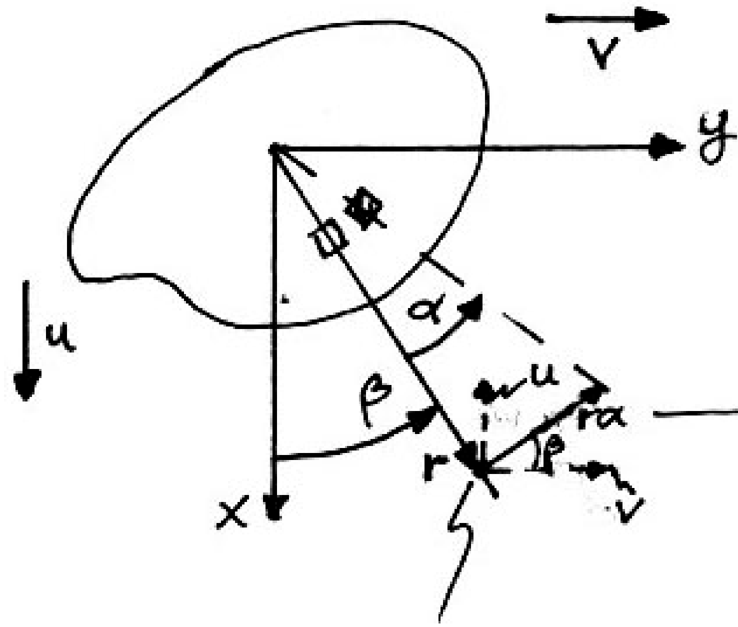
(careful! Rivello uses ϕ!)

rate of twist

a constant

(define as 0 @ z = 0)

Figure 10.2 **Representation of deformation of cross-section due to torsion**



(for small α)

undeformed position

This results in:

consider direction of + u

$$u\ (x, y, z) = r\alpha\ (-\sin\beta)$$

$$v\ (x, y, z) = r\alpha\ (\cos\beta)$$

$$w\ (x, y, z) = w\ (x, y)$$

$\Rightarrow$ **independent of z!**

We can see that:

$$r = \sqrt{x^2 + y^2}$$

$$\sin\beta = \frac{y}{r}$$

$$\cos\beta = \frac{x}{r}$$

This gives:

$$u\,(x, y, z) = -y\ k\ z \qquad (10 - 1)$$

$$v\,(x, y, z) = x\ k\ z \qquad (10 - 2)$$

$$w\,(x, y, z) = w\,(x, y) \qquad (10 - 3)$$

Next look at the <u>Strain-Displacement equations</u>:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = 0$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0$$

(<u>**consider**</u>: u exists, but $\frac{\partial u}{\partial x} = 0$

v exists, but $\frac{\partial v}{\partial y} = 0$)

$\Rightarrow$ No extensional strains in torsion <u>*if*</u> cross-sections are free to warp

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -z\ k + z\ k = 0$$

$\Rightarrow$ cross-section does not change shape (as assumed!)

$$\varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = k\,x + \frac{\partial w}{\partial y} \qquad (10-4)$$

$$\varepsilon_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = -k\,y + \frac{\partial w}{\partial x} \qquad (10-5)$$

Now the Stress-Strain equations:

let's first do ***isotropic***

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] = 0$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right] = 0$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] = 0$$

$$\Rightarrow \sigma_{xx},\ \sigma_{yy},\ \sigma_{zz} = 0$$

$$\varepsilon_{xy} = \frac{2(1+\nu)}{E}\sigma_{xy} = 0 \Rightarrow \sigma_{xy} = 0$$

$$\varepsilon_{yz} = \underbrace{\frac{2(1+\nu)}{E}}_{1/G}\sigma_{yz} \qquad (10-6)$$

$$\varepsilon_{xz} = \overbrace{\frac{2(1+\nu)}{E}}^{1/G}\sigma_{xz} \qquad (10-7)$$

$\Rightarrow$ only σ_{xz} and σ_{yz} stresses exist

Look at ***orthotropic*** case:

$$\varepsilon_{xx} = \frac{1}{E_{11}}\left[\sigma_{xx} - \nu_{12}\sigma_{yy} - \nu_{13}\sigma_{zz}\right] = 0$$

$$\varepsilon_{yy} = \frac{1}{E_{22}}\left[\sigma_{yy} - \nu_{21}\sigma_{xx} - \nu_{23}\sigma_{zz}\right] = 0$$

$$\varepsilon_{zz} = \frac{1}{E_{33}}\left[\sigma_{zz} - \nu_{31}\sigma_{xx} - \nu_{32}\sigma_{yy}\right] = 0$$

$\Rightarrow \sigma_{xx}, \sigma_{yy}, \sigma_{zz} = 0$ **still** equal zero

$$\varepsilon_{yz} = \frac{1}{G_{23}} \sigma_{yz}$$

$$\varepsilon_{xz} = \frac{1}{G_{13}} \sigma_{xz}$$

Differences are in ε_{yz} and ε_{xz} here as there are two <u>different</u> shear moduli (G_{23} and G_{13}) which enter in here.

for ***anisotropic material***:

coefficients of mutual influence and Chentsov coefficients foul everything up (no longer *"simple"* torsion theory). [can't separate torsion from extension]

Back to general case…

Look at the <u>Equilibrium Equations</u>:

$$\frac{\partial \sigma_{xz}}{\partial z} = 0 \qquad \Rightarrow \sigma_{xz} = \sigma_{xz}(x, y)$$

$$\frac{\partial \sigma_{yz}}{\partial z} = 0 \qquad \Rightarrow \sigma_{yz} = \sigma_{yz}(x, y)$$

So, σ_{xz} and σ_{yz} are only functions of x and y

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0 \qquad (10 - 8)$$

We satisfy equation (10 - 8) by introducing a Torsion (Prandtl) Stress Function ϕ (x, y) where:

$$\frac{\partial \phi}{\partial y} = -\sigma_{xz} \qquad (10 - 9a)$$

$$\frac{\partial \phi}{\partial x} = \sigma_{yz} \qquad (10 - 9b)$$

Using these in equation (10 - 8) gives:

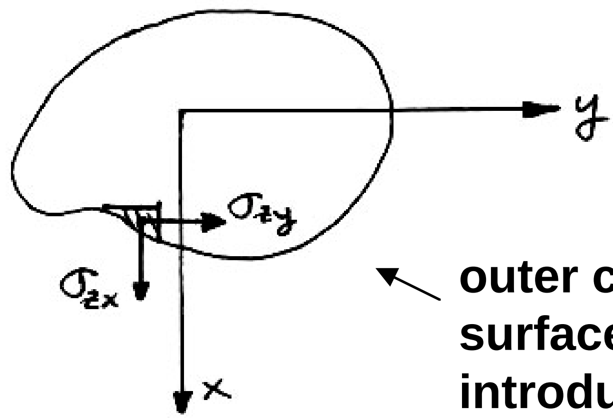
$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) \equiv 0$$

$\Rightarrow$Automatically satisfies equilibrium (as a stress function is supposed to do)

Now consider the <u>Boundary</u> <u>Conditions</u>:

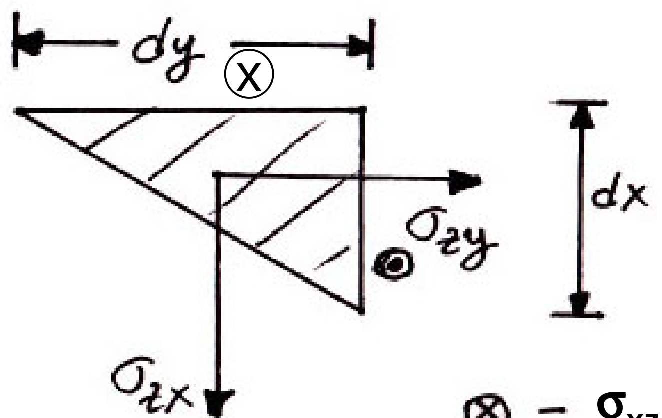
(a) Along the contour of the cross-section

Figure 10.3 **Representation of stress state along edge of solid cross-section under torsion**



outer contour is stress-free surface (away from load introduction)

Figure 10.4 **Close-up view of edge element from Figure 10.3**



⊗ – **σ_{xz} (into page)**

⊙ – **σ_{yz} (out of page)**

Using equilibrium:

$$\sum F_z = 0 \qquad \text{(out of page is positive)}$$

gives:

$$-\sigma_{xz}\, dydz + \sigma_{yz}\, dxdz = 0$$

Using equation (10 - 9) results in

$$-\left(-\frac{\partial \phi}{\partial y}\, dy\right) + \left(\frac{\partial \phi}{\partial x}\right) dx = 0$$

$$\left(\frac{\partial \phi}{\partial y}\, dy\right) + \left(\frac{\partial \phi}{\partial x}\, dx\right) = d\phi$$

And this means:

$d\phi = 0$

$\Rightarrow \phi = \text{constant}$

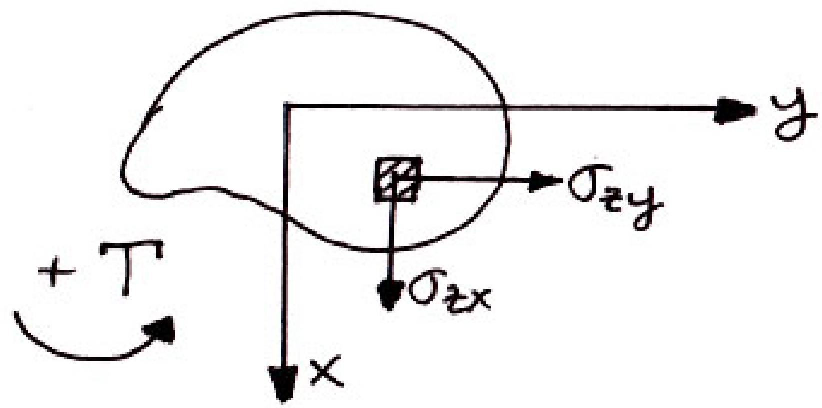
We take:

$$\boxed{\phi = 0} \text{ along contour} \qquad (10 - 10)$$

Note: addition of an arbitrary constant does not affect the stresses, so choose a convenient one (0!)

Boundary condition (b) on edge $z = l$

Figure 10.5 **Representation of stress state at top cross-section of rod under torsion**



Equilibrium tells us the force in each direction:

$$F_x = \iint \sigma_{zx} \, dxdy$$

using equation (10 - 9):

$$= \iint_{y_L}^{y_R} \frac{\partial \phi}{\partial y} \, dxdy$$

where y_R and y_L are the geometrical limits of the cross-section in the y direction

$$= -\int [\phi]_{y_L}^{y_R} dx$$

and since $\phi = 0$ on contour

$F_x = 0$ **O.K.** (since no force is applied in x-direction)

Similarly:

$$F_y = \int\int \sigma_{zy} \, dxdy = 0$$ **O.K.**

Look at one more case via equilibrium:

$$\text{Torque} = T = \int\int \left[x\sigma_{zy} - y\sigma_{zx} \right] dxdy$$

$$= \iint_{x_T}^{x_B} x \frac{\partial \phi}{\partial x} \, dxdy + \iint_{y_L}^{y_R} y \frac{\partial \phi}{\partial y} \, dydx$$

where x_T and x_B are geometrical limits of the cross-section in the x-direction

Integrate each term by parts:

$$\int AdB = AB - \int BdA$$

Set:

$$A = x \Rightarrow dA = dx$$

$$dB = \frac{\partial \phi}{\partial x}\, dx \Rightarrow B = \phi$$

and similarly for y

$$T = \int \Big[\underbrace{\big[x\phi\big]_{x_T}^{x_B}}_{\substack{= 0 \\ \text{since } \phi = 0 \text{ in contour}}} - \int \phi dx \Big] dy + \int \Big[\underbrace{\big[y\phi\big]_{y_L}^{y_R}}_{\substack{= 0 \\ \text{since } \phi = 0 \text{ in contour}}} - \int \phi dy \Big] dx$$

$$\Rightarrow \boxed{T = -2 \iint \phi \, dxdy} \qquad (10 - 11)$$

Up to this point, all the equations [with the slight difference in stress-strain of equations (10 - 6) and (10 - 7)] are also valid for orthotropic materials.

Summarizing

- Long, prismatic bar under torsion
- Rate of twist, k = constant
- $\varepsilon_{yz} = kx + \frac{\partial w}{\partial y}$
- $\varepsilon_{xz} = - ky + \frac{\partial w}{\partial x}$
- $\frac{\partial \phi}{\partial y} = -\sigma_{xz}$ $\qquad$ $\frac{\partial \phi}{\partial x} = \sigma_{yz}$
- Boundary conditions

$\phi = 0$ on contour (free boundary)

$$T = -2 \iint \phi \, dxdy$$

Solution of Equations

(now let's go back to isotropic)

Place equations (10 - 4) and (10 - 5) into equations (10 - 6) and (10 - 7) to get:

$$\sigma_{yz} = G\varepsilon_{yz} = G\left(k\,x + \frac{\partial w}{\partial y}\right) \qquad (10 - 12)$$

$$\sigma_{xz} = G\varepsilon_{xz} = G\left(-k\,y + \frac{\partial w}{\partial x}\right) \qquad (10 - 13)$$

We want to eliminate w. We do this via:

$$\frac{\partial}{\partial x}\{\text{Eq. } (10 - 12)\} - \frac{\partial}{\partial y}\{\text{Eq. } (10 - 13)\}$$

to get:

$$\frac{\partial \sigma_{yz}}{\partial x} - \frac{\partial \sigma_{xz}}{\partial y} = G\left(k + \frac{\partial^2 w}{\partial x \partial y} + k - \frac{\partial^2 w}{\partial y \partial x}\right)$$

and using the definition of the stress function of equation (10 - 9) we get:

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2Gk} \qquad (10 - 14)$$

Poisson's Equation for ϕ

(Nonhomogeneous Laplace Equation)

Note for orthotropic material

We do **not** have a common shear modulus, so we would get:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \left(G_{xz} + G_{yz}\right) k + \left(G_{yz} - G_{xz}\right) \frac{\partial^2 w}{\partial x \partial y}$$

$\Rightarrow$ We cannot eliminate w unless G_{xz} and G_{yz} are virtually the same

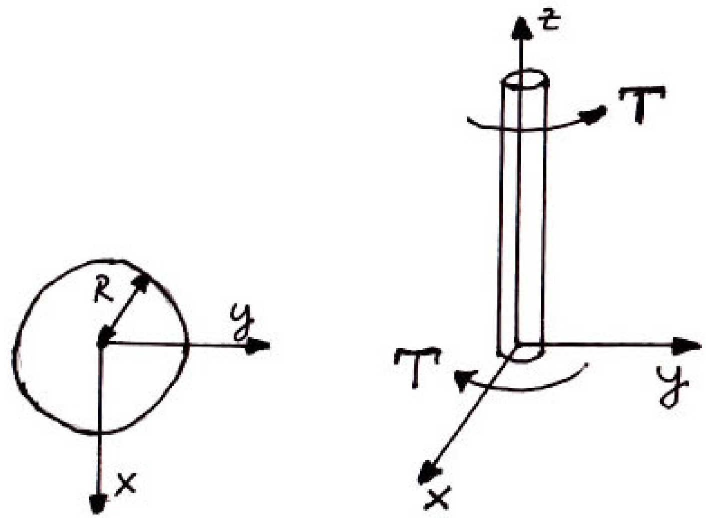
Overall solution procedure:

- Solve Poisson equation (10 - 14) subject to the boundary condition of $\phi = 0$ on the contour
- Get T - k relation from equation (10 - 11)
- Get stresses (σ_{xz}, σ_{yz}) from equation (10 - 9)
- Get w from equations (10 - 12) and (10 - 13)
- Get u, v from equations (10 - 1) and (10 - 2)
- Can also get ε_{xz}, ε_{yz} from equations (10 - 6) and (10 - 7)

This is *"St. Venant Theory of Torsion"*

Application to a Circular Rod

Figure 10.6 **Representation of circular rod under torsion cross-section**



"Let":

$$\phi = C_1 (x^2 + y^2 - R^2)$$

This satisfies $\phi = 0$ on contour since $x^2 + y^2 = R^2$ on contour

This gives:

$$\frac{\partial^2 \phi}{\partial x^2} = 2C_1 \qquad \frac{\partial^2 \phi}{\partial y^2} = 2C_1$$

Place these into equation (10-14):

$$2C_1 + 2C_1 = 2Gk$$

$$\Rightarrow C_1 = \frac{Gk}{2}$$

<u>Note</u>: (10-14) is satisfied <u>exactly</u>

Thus:

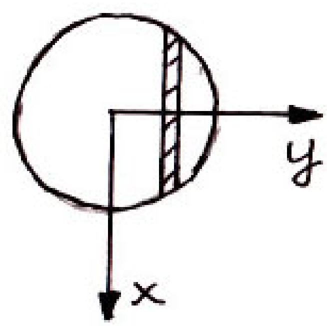
$$\boxed{\phi = \frac{Gk}{2}\left(x^2 + y^2 - R^2\right)}$$

Satisfies boundary conditions and partial differential equation exactly

Now place this into equation (10-11):

$$T = -2 \iint \phi \, dxdy$$

Figure 10.7 **Representation of integration strip for circular cross-section**



$$T = Gk \int_{-R}^{R} \int_{-\sqrt{R^2 - y^2}}^{+\sqrt{R^2 - y^2}} \left(R^2 - y^2 - x^2\right) dxdy$$

$$T = Gk \int_{-R}^{R} \left[\left(R^2 - y^2\right) x - \frac{x^3}{3} \right]_{-\sqrt{R^2 - y^2}}^{+\sqrt{R^2 - y^2}} dy$$

$$= Gk \frac{4}{3} \int_{-R}^{R} \left(R^2 - y^2\right)^{3/2} dy$$

$$= Gk \frac{4}{3} \frac{1}{4} \left[\underbrace{y\left(R^2 - y^2\right)^{3/2}}_{= 0} + \underbrace{\frac{3}{2} R^2 y \sqrt{R^2 - y^2}}_{= 0} + \underbrace{\frac{3}{2} R^4 \sin^{-1} \frac{y}{R}}_{= \frac{3}{2} R^4 \pi} \right]_{-R}^{+R}$$

This finally results in

$$T = Gk \frac{\pi R^4}{2}$$

Since k is the rate of twist: $k = \dfrac{d\alpha}{dz}$, we can rewrite this as:

$$\boxed{\frac{d\alpha}{dz} = \frac{T}{GJ}}$$

where:

J = torsion constant $\left(= \dfrac{\pi R^4}{2} \text{ for a circle} \right)$

α = amount of twist

and:

GJ = torsional rigidity

<u>Note</u> similarity to:

$$\frac{d^2w}{dx^2} = \frac{M}{EI}$$

where: EI = bending rigidity

(I) J - geometric part

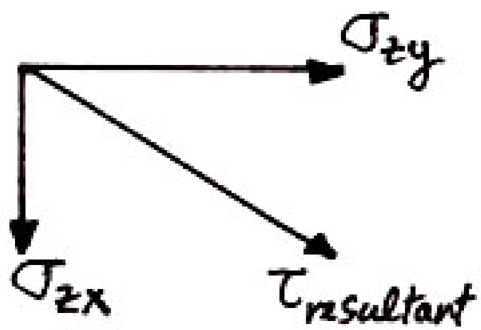
(E) G - material part

To get the stresses, use equation (10 - 9):

$$\sigma_{yz} = \frac{\partial\phi}{\partial x} = Gkx = \frac{T}{J}x$$

$$\sigma_{xz} = -\frac{\partial\phi}{\partial y} = -Gky = -\frac{T}{J}y$$

Figure 10.8 **Representation of resultant shear stress, τ_{res}, as defined**



Define a resultant stress:

$$\tau = \sqrt{\sigma_{zx}^2 + \sigma_{zy}^2}$$

$$= \frac{T}{J}\underbrace{\sqrt{x^2 + y^2}}_{= r}$$

The final result is:

$$\boxed{\tau = \frac{Tr}{J}}$$

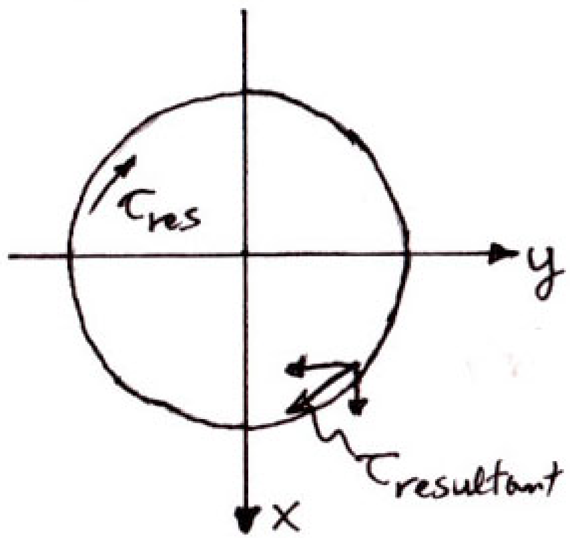
for a circle

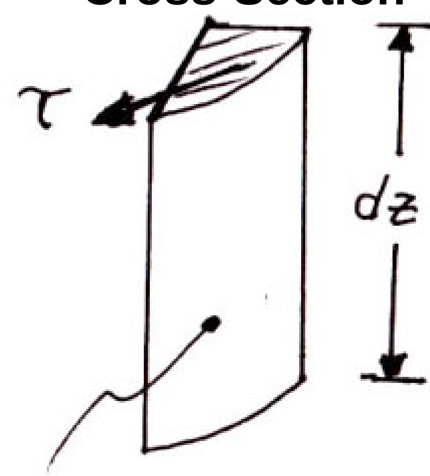
Note: similarity to $\left(\sigma_x = -\frac{Mz}{I}\right)$

τ <u>always</u> acts along the contour (shape)

resultant

Figure 10.9 **Representation of shear resultant stress for circular cross-section**

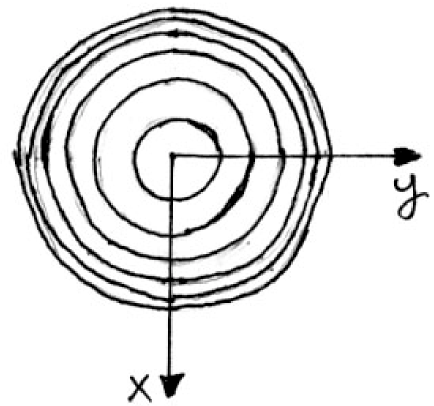




Also note:

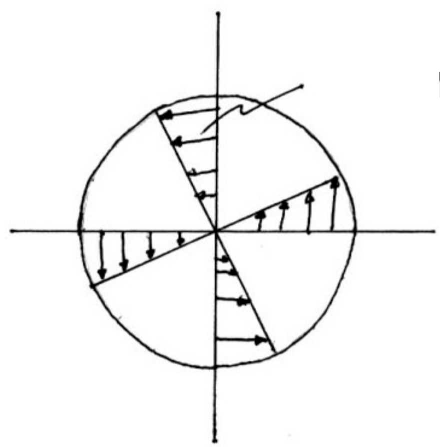
1. Contours of ϕ: close together near edge $\Rightarrow$ higher τ

Figure 10.10 **Representation of contours of torsional shear function**



2. Stress pattern (τ) creates twisting

Figure 10.11 **Representation of shear stresses acting perpendicular to radial lines**



To get the deflections, first find α:

$$\frac{d\alpha}{dz} = \frac{T}{GJ}$$

(pure rotation of cross-section)

integration yields:

$$\alpha = \frac{Tz}{GJ} + C_1$$

Let $C_1 = 0$ by saying $\alpha = 0$ @ $z = 0$

Use equations (10 - 1) and (10 - 2) to get:

$$u = -yzk = -y\frac{Tz}{GJ}$$

$$v = xzk = x\frac{Tz}{GJ}$$

Go to equations (10 - 12) and (10 - 13) to find w(x, y):

Equation (10 - 12) gives:

$$\frac{\partial w}{\partial y} = \frac{\sigma_{yz}}{G} - kx$$

using the result for σ_{yz}:

$$\frac{\partial w}{\partial y} = \frac{Gkx}{G} - kx = 0$$

integration of this says

$w(x, y) = g_1\,(x)$ *(not a function of y)*

In a similar manner…

Equation (10 -13) gives:

$$\frac{\partial w}{\partial x} = \frac{\sigma_{xz}}{G} + ky$$

Using σ_{xz} = -Gky gives:

$$\frac{\partial w}{\partial x} = -\frac{Gky}{G} + ky = 0$$

integration tells us that:

$w(x, y) = g_2\,(y)$ *(not a function of x)*

Using these two results we see that if w(x, y) is neither a function of x nor y, then it must be a constant. Might as well take this as **zero**

(other constants just show a rigid displacement in z which is trivial)

$\Rightarrow$ $w(x, y) = 0$ **No warping for circular cross-sections**

(this is the only cross-section that has no warping)

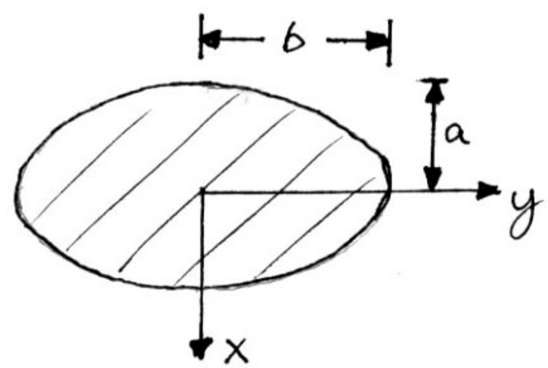
Other Cross-Sections

In other cross-sections, warping is “the ability of the cross-section to resist torsion by differential bending”.

2 parts for torsional rigidity

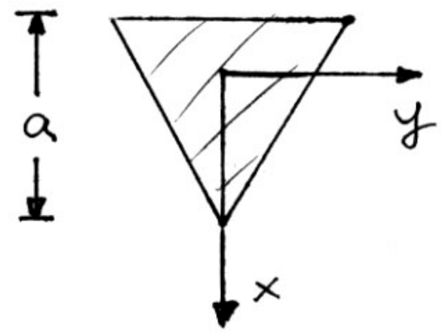
- Rotation
- Warping

Ellipse



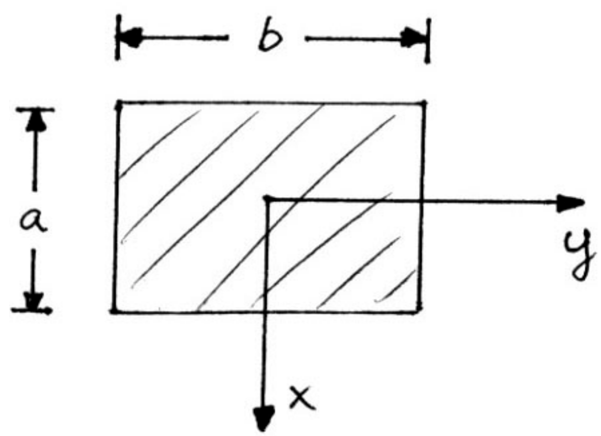
$$\phi = C_1\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

Equilateral Triangle



$$\phi = C_1\left(x - \sqrt{3}y + \frac{2}{3}a\right)\left(x + \sqrt{3}y - \frac{2}{3}a\right)\left(x + \frac{1}{3}a\right)$$

Rectangle



$$\phi = \sum_{n\ odd} \left(C_n + D_n \cosh \frac{n\pi y}{b} \right) \cos \frac{n\pi x}{a}$$

Series: (the more terms you take, the better the solution)

These all give solutions to $\nabla^2 \phi = 2GK$ subject to $\phi = 0$ on the boundary. In general, there *will* be warping

see Timoshenko for other relations (Ch. 11)

Note: there are also solutions via "warping functions". This is a displacement formulation

see Rivello 8.4

Next we'll look at an analogy used to "solve" the general torsion problem