Nonlinear algorithms

IVP

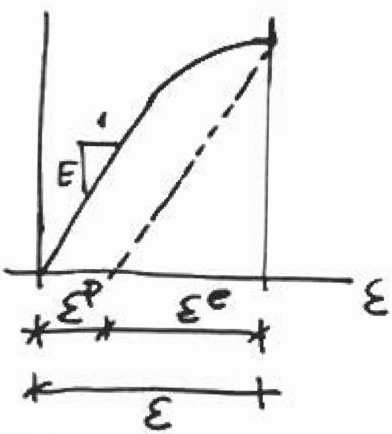
$$A\dot{y} + B(y) = b(t)$$

Small-Strain Plasticity

Ref.: "Plasticity theory", Lubliner, J.
Macmillan (1990)

"Computational Inelasticity", Simo, J and Hughes, T.J.R. - Springer-Verlag (1998)

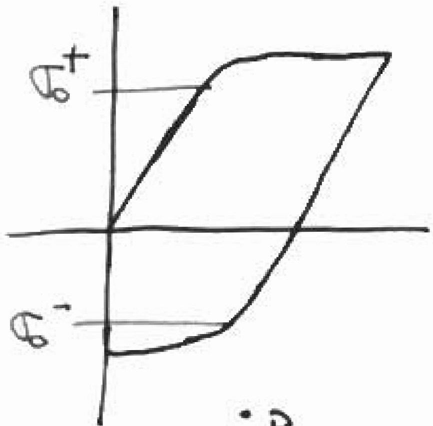
Phenomenology of (metal) plasticity: Uniaxial



$$\varepsilon^e = \frac{\sigma}{E} \quad , \quad \boxed{\varepsilon^p = \varepsilon - \varepsilon^e}$$

↳ Hooke's Law

$$\boxed{\sigma = E(\varepsilon - \varepsilon^p) = E\,\varepsilon^e}$$



$-\sigma_0^- \neq \sigma_0^+$

Bauschinger effect

if $\sigma_0^- < \sigma < \sigma_0^+ \Rightarrow \dot{\varepsilon}^p = 0$

$\dot{\varepsilon}^p \neq 0$ if $\sigma = \sigma_0^-$, $\dot{\sigma} < 0$
$\sigma = \sigma_0^+$, $\dot{\sigma} > 0$

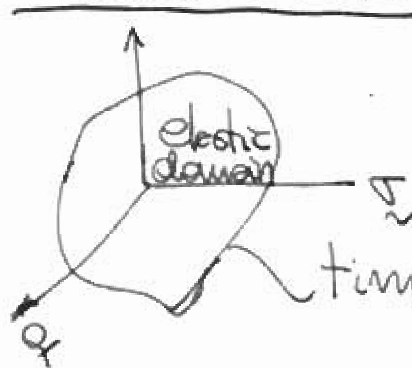
$\dot{\varepsilon}^p = 0$ if $\left.\begin{array}{l} \sigma = \sigma_0^- , \ \dot{\sigma} > 0 \\ \sigma = \sigma_0^+ , \ \dot{\sigma} < 0 \end{array}\right\}$ elastic unloading

Several dimensions

- **kinematics:** $\varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^p_{ij}$
- Define decomposition by elastic unloading

$$\varepsilon^e_{ij} = C^{-1}_{ijkl}\,\sigma_{kl} \quad , \quad \sigma_{ij} = C_{ijkl}\,\varepsilon^e_{kl}$$

- Elastic domain: all stress paths within elastic domain are elastic ($\dot{\varepsilon}^p = 0$)



time dependent.

- Flow rule: defines direction of plastic strain increment

$$\boxed{\dot{\varepsilon}^{p}_{ij} = \dot{\lambda}\, r_{ij}(\underset{\sim}{\sigma}, \underset{\sim}{q})}$$

$\underset{\sim}{q} \equiv$ internal variables (model dependent)

- Viscosity law gives magnitud of $\dot{\varepsilon}^{p}_{ij}$

$$\boxed{\dot{\lambda} = \frac{\phi(\sigma, q)}{\eta}}$$

$\phi \equiv$ effective stress

$\eta \equiv$ viscosity parameter

$[\eta] \equiv \dfrac{\text{stress}}{\text{strain rate}}$

- Elastic domain: $\phi(\sigma, q) \leq 0$

- Kinetic equations: hardening laws

$$\boxed{\dot{q}_{\alpha} = f_{\alpha}(\sigma, q) = \dot{\lambda}\, h_{\alpha}(\sigma, q)}$$

$h_{\alpha} \equiv$ hardening moduli

when $\dot{\lambda}=0$, $\dot{\varepsilon}^p=0$, $\dot{q}_\alpha=0$ → elastic (reversible response)

Rate-independent behavior: inviscid limit $\eta \to 0$

$$\dot{\varepsilon}^p_{ij} = \dot{\lambda}\, r_{ij} \qquad \dot{\lambda} = \frac{\phi}{\eta} \qquad \phi \geq 0$$

if $\eta \to 0$, for $\dot{\lambda} < \infty \implies \boxed{\phi = 0}$ during plastic flow

yield condition

$\phi \equiv$ overstress (plastic)

Viscocity law:

$$\dot{\lambda} = \begin{cases} \dfrac{\phi}{\eta} & \text{if } \phi \geq 0 \\ 0 & \text{if } \phi \leq 0 \end{cases}$$

allows to determine $\dot{\lambda}$ in rate-dependent case

Rate-independent:

- $\phi = 0$ for plastic flow
- $\dot{\lambda}$ cannot be determined from viscosity law

- $\dot{\lambda}$ determined from constraint $\phi = 0$

$\Longrightarrow$ Loading- unloading conditions (Kuhn-Tucker form):

The following three conditions must be satisfied at all times:

$$\begin{cases} ① & \phi \leq 0 \\ ② & \dot{\lambda} \geq 0 \quad \text{(irreversibility)} \\ ③ & \dot{\lambda}\phi = 0 \end{cases}$$

Case ⓐ:

$$\phi < 0 \Rightarrow \dot{\lambda} = 0 \Rightarrow \dot{\varepsilon}_{ij}^{p} = 0, \; \dot{q}_{\alpha} = 0$$

case ⓑ: $\dot{\lambda} > 0 \Rightarrow \phi = 0$ (for plastic flow to occur, yield condition must be satisfied)

Summary of small-strain plasticity

	Rate-dependent	Rate-independent
• Hooke's Law	$\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^{p})$	same

flow rule $\qquad \dot{\varepsilon}^{p}_{ij} = \dot{\lambda}\, r_{ij}(\sigma, q) \quad (\|r\| = 1) \qquad$ same

hardening $\qquad \dot{q}_{\alpha} = \dot{\lambda}\, h_{\alpha}(\sigma, q) \qquad$ same

viscosity law $\qquad \dot{\lambda} = \begin{cases} \phi(\sigma, q)/\eta & \text{if } \phi \geq 0 \\ 0 & \text{if } \phi < 0 \end{cases} \qquad$ $\phi \leq 0,\ \dot{\lambda} \geq 0$, $\phi \dot{\lambda} = 0$ (loading/unloading conditions)

For "associated" flow rule:

$$\boxed{r_{ij} = \frac{\partial \phi}{\partial \sigma_{ij}}} \qquad \text{(normality)}$$

Elastic-plastic moduli

Relation between $\dot{\sigma}_{ij}$, $\dot{\varepsilon}_{kl}$

Hooke's law: $\dot{\sigma}_{ij} = C_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}^{p}_{kl})$

Plastic hardening: $\phi(\sigma, q) = 0$

$$\dot{\phi} = 0 = \underbrace{\frac{\partial \phi}{\partial \sigma_{ij}}}_{\nu_{ij}} \dot{\sigma}_{ij} + \underbrace{\frac{\partial \phi}{\partial q_{\alpha}}}_{s_{\alpha}} \dot{q}_{\alpha}$$

$$\frac{\partial \phi}{\partial \sigma_{ij}} = \nu_{ij}(\sigma, q) \quad , \quad \frac{\partial \phi}{\partial q_\alpha} = \xi_\alpha(\sigma, q)$$

$$0 = \nu_{ij}\, C_{ijkl}\left(\dot{\varepsilon}_{kl} - \dot{\lambda}\, r_{kl}\right) + \xi_\alpha \dot{\lambda}\, h_\alpha \Longrightarrow$$

$$\dot{\lambda} = \frac{\nu_{ij}\, C_{ijkl}\, \dot{\varepsilon}_{kl}}{\nu_{ij} C_{ijkl}\, r_{kl} - \xi_\alpha h_\alpha}$$

$$\dot{\sigma}_{ij} = C_{ijkl}\left(\dot{\varepsilon}_{kl} - \dot{\lambda}\, r_{kl}\right)$$

$$= C_{ijkl}\dot{\varepsilon}_{kl} - C_{ijkl}\left(\frac{\nu_{pq}\, C_{pqrs}\, \dot{\varepsilon}_{rs}}{\nu_{mn}\, C_{mnpq}\, r_{pq} - \xi_\alpha h_\alpha}\right) r_{kl}$$

$$\dot{\sigma}_{ij} = \underbrace{\left(C_{ijkl} - \frac{C_{ijmn}\, r_{mn}\, \nu_{pq}\, C_{pqkl}}{\nu_{mn}\, C_{mnpq}\, r_{pq} - \xi_\alpha h_\alpha}\right)}_{C_{ijkl}^{ep} \,\equiv\, \text{elastoplastic tangent moduli}} \dot{\varepsilon}_{kl}$$

$$\Longrightarrow \quad \dot{\sigma}_{ij} = C_{ijkl}\, \dot{\varepsilon}_{kl} \quad \text{(elastic unloading)}$$

$$\dot{\sigma}_{ij} = C_{ijkl}^{ep}\, \dot{\varepsilon}_{kl} \quad \text{(plastic loading)}$$

$$\boxed{C^{ep} = C - \frac{(C:r)\otimes(\nu:C)}{(\nu:C):r - \xi\cdot h}}$$

$a:b = a_{ij}\, b_{ij}$

$a\otimes b = a_{ij}\, b_{kl}$

Examples : J$_2$-flow theory, isotropic, power-law hardening, power-law viscosity

$$\dot{\lambda} = \begin{cases} \dot{\varepsilon}_0 \left[\left(\dfrac{\bar{\sigma}}{\sigma_0} \right)^m - 1 \right] & \text{if } \bar{\sigma} \geq \sigma \\ 0 & \text{if } \bar{\sigma} \leq \sigma \end{cases}$$

ε_0, m constants

$\bar{\sigma}$: Mises stress

$$\bar{\sigma} = \left[\frac{3}{2} s_{ij}\, s_{ij} \right]^{1/2}$$

$$s_{ij} = \sigma_{ij} - p\,\delta_{ij} \quad , \quad p = \frac{\sigma_{kk}}{3}$$

Hardening:

$$\sigma_0 = \sigma_y \left(1 + \frac{\bar{\varepsilon}}{\varepsilon_0} \right)^{1/n} ; \quad \bar{\varepsilon} = \int_0^t \dot{\lambda}\, dt$$

$\sigma_y, \varepsilon_0, n$: constants

Flow rule: $$\boxed{\dot{\varepsilon}_{ij}^{p} = \dot{\lambda} \, \frac{3}{2} \, \frac{s_{ij}}{\bar{\sigma}}}$$ Prandtl-Reuss flow rule

Then $$\dot{\bar{\varepsilon}} = \left(\frac{2}{3} \dot{\varepsilon}_{ij}^{p} \dot{\varepsilon}_{ij}^{p} \right)^{1/2}$$

$$\{ q \} = \{ \bar{\varepsilon} \} \quad , \quad r_{ij} = \frac{3}{2} \frac{s_{ij}}{\bar{\sigma}}$$

$$\eta = \frac{\sigma_y}{\dot{\varepsilon}_0} \rightarrow \dot{\lambda} = \frac{1}{\eta} \underbrace{\sigma_y \left[\left(\frac{\bar{\sigma}}{\sigma_0} \right)^m - 1 \right]}_{\phi} = \frac{\phi}{\eta}$$

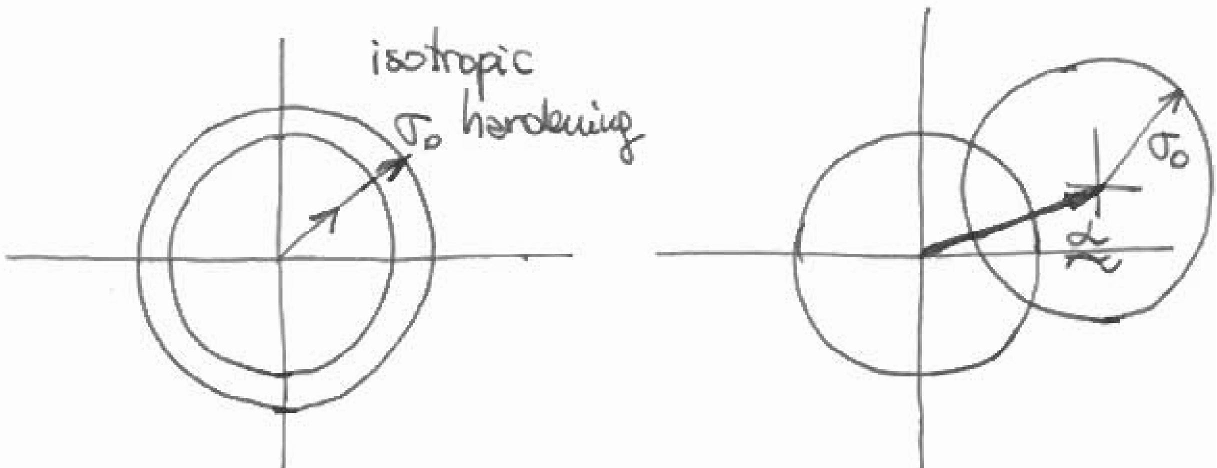
$$\phi = \sigma_y \left[\left(\frac{\bar{\sigma}}{\sigma_0} \right)^m - 1 \right]$$

$$\eta \rightarrow 0 \ , \ \phi = 0 \qquad \boxed{\left(\frac{\bar{\sigma}}{\sigma_0} \right)^m - 1 = 0}$$ Mises yield criterion

⊛ Check this is an associated flow rule, i.e.,

$$r_{ij} = \frac{\partial \phi}{\partial \sigma_{ij}}$$

Isotropic - kinematic hardening



Isotropic:

$$\phi = \frac{\sigma_y}{\eta}\left[\left(\frac{\bar{\sigma}}{\sigma_o}\right)^m - 1\right], \quad \bar{\sigma} = \sqrt{\frac{3}{2} s_{ij} s_{ij}}$$

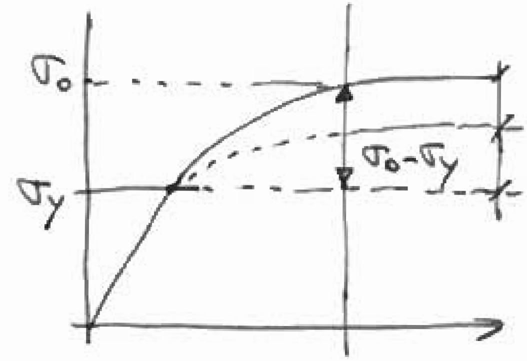
$$\sigma_o = \underbrace{\sigma_y \left(1 + \frac{\bar{\varepsilon}}{\varepsilon_o}\right)^{1/n}}_{\tilde{\sigma}_o(\bar{\varepsilon})}$$

Isotropic-kinematic

$$\phi = \frac{\sigma_y}{\eta}\left[\left(\frac{\bar{\sigma}}{\sigma_o}\right)^m - 1\right], \quad \bar{\sigma} = \sqrt{\frac{3}{2} \bar{s}_{ij} \bar{s}_{ij}}$$

$$\bar{s}_{ij} = s_{ij} - \alpha_{ij}$$

where $\alpha_{ij} = C(\bar{\varepsilon})\, \underset{\sim}{\varepsilon}^p$



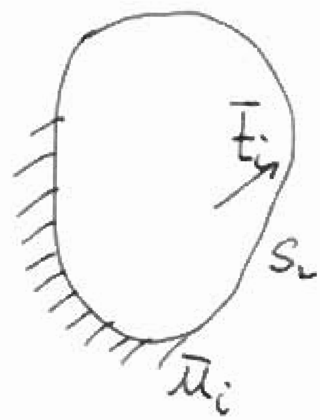
$$\sigma_0 - \sigma_y = (1-\beta)\left[\tilde{\sigma}_0(\bar{\varepsilon}) - \sigma_y\right]$$

$$\alpha_{ij} = C(\bar{\varepsilon})\, \varepsilon_{ij}^p \quad , \quad C(\bar{\varepsilon}) = \beta\, \frac{\tilde{\sigma}_0(\bar{\varepsilon}) - \sigma_y}{\bar{\varepsilon}}$$

Boundary value problem

$$\begin{cases} \sigma_{ij,j} + f_i = 0 & \text{in } B \\ \sigma_{ij}\, n_j = \bar{t}_i & \text{on } S_2 \end{cases}$$

$$\begin{cases} u_i = \bar{u}_i & \text{on } S_1 \\ \varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) & \text{in } B \end{cases}$$



$\varepsilon^e, \varepsilon^p$ incompatible

Variational principle (minimum potential energy)