79A

To obtain pressures: Define "M_p" shape functions

$$M_p(\bar{\xi}_q) = \delta_{pq}$$

Then $p^e(x) = \sum_{q=1}^{\bar{Q}} p_q M_q(x)$

where:

$$p_q = \cancel{k}\, \theta(\bar{\xi}_q)$$

Constraint count: # dof/element $= 2 \frac{1}{4} 4 = 2$

constraints/element $= 1$

$\boxed{r=2}$ optimal!

2) Assumed strain methods

Objective: to control number of (volumetric)

constraints independently → Independent

interpolations for displacements and strain.

Need <u>two-field</u> variational formulation. Start from Hu-Washizu.

$$J(u,\varepsilon,\sigma) = \int_B \left[W(\varepsilon) + \sigma_{ij}(u_{(i,j)} - \varepsilon_{ij}) - f_i u_i \right] dV -$$

$$- \int_{S_1} \sigma_{ij} n_j (u_i - \bar{u}_i)\, ds - \int_{S_2} \bar{t}_i u_i \, ds$$

- Assume constitutive law is satisfied: $\sigma_{ij} = \dfrac{\partial W}{\partial \varepsilon_{ij}}$
- Assume displacement boundary conditions are satisfied:

$$u_i = \bar{u}_i \text{ on } S_1$$

- Assume linear elasticity: $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$

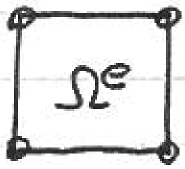
$$J(u,\varepsilon) = \int_B \left[\frac{1}{2} C_{ijkl}\,\varepsilon_{ij}\,\varepsilon_{kl} + C_{ijkl}\,\varepsilon_{kl}(u_{(i,j)} - \varepsilon_{ij}) - f_i u_i \right] dV$$

$$- \int_{S_2} \bar{t}_i u_i \, ds$$

Hellinger-Reissner

$$\boxed{J(u,\varepsilon) = \int_B \left[-\frac{1}{2} C_{ijkl}\,\varepsilon_{kl}\,\varepsilon_{ij} + C_{ijkl}\,\varepsilon_{kl}\,u_{(i,j)} - f_i u_i \right] dV - \int_{S_2} \bar{t}_i u_i \, ds}$$

Introduce independent interpolation for u, ε:



$$u_i^e = \sum_{a=1}^{n} u_{ia} N_a^e$$

$$u^e_{(i,j)} \equiv B(x)\, u \qquad \neq \varepsilon(x)$$

$$\varepsilon_h(x) = B^{dev}(x)\, u + \varepsilon_h^{vol}$$

$$\varepsilon_h^{vol} = \text{const.}$$

$\rightarrow$ 1 volumetric constraint/element

Discretized functional

$$\boxed{\begin{aligned} J_h(u_h, \varepsilon_h) = \sum_e \int_{\Omega^e} \Big[& u^T B^T C \left(B^{dev} u + \varepsilon_h^{vol}\right) - \\ & - \frac{1}{2} \left(B^{dev} u + \varepsilon_h^{vol}\right)^T C \left(B^{dev} u + \varepsilon_h^{vol}\right) dV \\ & - FT \end{aligned}}$$

Euler equations

(a) $\langle DJ_h(u_h, \varepsilon_h), \eta_h \rangle = 0 \rightarrow$ equilibrium

(b) $\langle DJ_h(u_h, \varepsilon_h), \alpha_h \rangle = 0 \rightarrow$ compatibility

(a) $$\sum_e \int_{\Omega^e} \eta^T B^T C \left(B^{dev} u + \varepsilon_h^{vol}\right) + \eta^T \left(B^{dev}\right)^T C B u$$

$$-\eta^T (B^{dev})^T C (B^{dev} u + \varepsilon_h^{vol})\, dV - FT = 0$$

(b) $$\sum_{e=1}^{E} \int_{\Omega^e} (\alpha_h^{vol})^T \left[C B u - C (B^{dev} u + \varepsilon_h^{vol}) \right] dV = 0$$

decouples into "E" independent systems

$$(\alpha_h^{vol})^T C \int_{\Omega^e} \Big[\underbrace{(B - B^{dev})}_{B^{vol}} u - \varepsilon_h^{vol} \Big] dV = 0$$

$$\int_{\Omega^e} \varepsilon_h^{vol}\, dV = V(\Omega^e)\, \varepsilon_h^{vol} = \left[\int_{\Omega^e} B^{vol}\, dV \right] u$$

$$\Longrightarrow \quad \boxed{\varepsilon_h^{vol} = \left[\frac{1}{V(\Omega^e)} \int_{\Omega^e} B^{vol} \right] u}$$

Mean dilatation method (Nagtegaal, Parks & Rice)

$$\boxed{\varepsilon_h^{vol} = \bar{B}^{vol} u, \quad \bar{B}^{vol} = \frac{1}{V(\Omega^e)} \int_{\Omega^e} B^{vol}\, dV}$$

$$\varepsilon_h(x) = B^{dev}_{(x)} u + \varepsilon_h^{vol} = \underbrace{\left(B^{dev}_{(x)} + \bar{B}^{vol} \right)}_{\bar{B}} u$$

$$\boxed{\varepsilon_h(x) = \bar{B} \, u}$$

Euler equations:

(a) $$\sum_e \int_{\Omega^e} \Big[\eta^T B^T C \, \bar{B} u + \eta^T (B^{dev})^T C B u -$$

$$- \eta^T (B^{dev})^T C \, \bar{B} \Big] dV - FT = 0$$

$$\sum_e \eta^T \underbrace{\int_{\Omega^e} \Big[B^T C \, \bar{B} + (B^{dev})^T C (B - \bar{B}) dV \Big]}_{K^e} u - FT = 0$$

$$K^e = \int_{\Omega^e} (\bar{B} + B - \bar{B})^T C \bar{B} + (B^{dev})^T C (B - \bar{B}) dV$$

$$\boxed{K^e = \int_{\Omega^e} \bar{B} C \bar{B} + (B - \bar{B})^T C \bar{B} + (B^{dev})^T C (B - \bar{B}) dV}$$

$$\underbrace{B - \bar{B}} = B^{dev} + B^{vol} - \left(B^{dev} + \bar{B}^{vol} \right) = \underbrace{B^{vol} - \bar{B}^{vol}}$$

General expression for anisotropic elasticity

Can be simplified in the case of <u>isotropic elasticity</u>:

$$\int_{\Omega^e} (B^{dev})^T C (B-\bar{B})\,dV = \int_{\Omega^e} (B^{dev})^T \underbrace{C\,(B^{vol}-\bar{B}^{vol})}_{\text{volumetric tensor}}\,dV$$

$$= 0$$

$$\int_{\Omega^e} \bar{B}^{vol} C (B-\bar{B})\,dV = \underbrace{\bar{B}^{vol} C}_{\text{constant}} \Big[\underbrace{\int_{\Omega^e} B^{vol}\,dV}_{V(\Omega)\,\bar{B}^{vol}} - \int_{\Omega^e} \bar{B}^{vol}\,dV \Big]$$

$$= 0$$

$$\boxed{K^e = \int_{\Omega^e} \bar{B}^T C \bar{B}\,dV}$$ Simo & Hughes (1985)

variationally consistent only if:

$$\boxed{\int_{\Omega^e} (B-\bar{B})^T C \bar{B}\,dV = 0}$$ ORTHOGONALITY CONDITION

<u>Example</u>: 4-node quadrilateral, constant C

with/without distortion $\Rightarrow$ $\boxed{\begin{array}{c}\text{ASSUMED STRAIN } \bar{B} \\ \equiv \\ \text{RIP}\end{array}}$

3) Mixed-methods (u, p)

Lagrangian for incompressible elastic solid:

$$L(u,p) = \int_B \left(\frac{1}{2} C_{ijkl}\, \varepsilon_{ij}^{dev}\, \varepsilon_{kl}^{dev} - f_i u_i \right) dV - \int_{S_2} t_i u_i\, dV + \int_B p\, u_{i,i}\, dV$$

$$L(u,p) = J(u) + \int_B p\, u_{i,i}\, dV$$

unconstrained potential

Introduce different interpolations for "u", "p"

$$u_{h_i}(x) = \sum_e \sum_a u_{ia}^e N_a^e \quad \leftarrow C^0 \text{ across element boundaries}$$

$$\boxed{p_h(x) = \sum_e \sum_\alpha p_\alpha^e M_\alpha^e(x)} \quad \leftarrow \text{need not be } C^0 \text{ across elements}$$

$$p_h^e(x) = \sum_\alpha p_\alpha^e \, M_\alpha^e(x) \longrightarrow \text{local element interpolation}$$

Examples: $M_\alpha \in P_k \equiv$ sets of polynomials up to order "k"

$$P_0 = \{1\}, \quad P_1 = \{1, x, y\}$$

Insert into L:

$$\boxed{L_h(u_h, p_h) = J_h(u_h) + p_h^T \, B_h^T \, u_h}$$

DISCRETIZED LAGRANGIAN

$$J_h(u_h) = \frac{1}{2} \, u_h^T \, K_h \, u_h - f^T u_h$$

$$K_h = \sum_e K_h^e \; ; \; K_h^e = \int_{\Omega^e} \left(B^{dev}\right)^T C \, B^{dev} \, dV$$

$$\boxed{B_h = \sum_e \int_{\Omega^e} M_\alpha^e \, N_{a,i}^e \, dV}$$

Euler equations for $L_h(u_h, p_h)$:

$$u_h \rightarrow \quad K_h u_h + B_h p_h = f_h$$

$$p_h \rightarrow \quad B_h^T u_h = 0$$

$$\left[\begin{array}{c|c} K_h & B_h \\ \hline B_h^T & 0 \end{array}\right] \left\{\begin{array}{c} u_h \\ \hline p_h \end{array}\right\} = \left\{\begin{array}{c} f_h \\ \hline 0 \end{array}\right\}$$

B_h: discretized gradient

B_h^T: discretized divergence

N: displacement dof; $\quad u_h \in \mathbb{R}^N \equiv V_h$

M: pressure dof; $\quad p_h \in \mathbb{R}^M \equiv Q_M$

$$\boxed{\dim B_h = N \times M}$$

In order to eliminate the pressure at the element level:

$$L_\varepsilon(u,p) = L(u,p) + \frac{\varepsilon}{2}\int_B p^2\, dV \; ; \quad \varepsilon \to 0$$

$$L_{\varepsilon h}(u,p) = L_h(u_h, p_h) - \frac{\varepsilon}{2}\, p_h^T M_h p_h$$

where $M_h = \sum_e \int_{\Omega^e} M_\alpha^e M_\beta^e \, dV$ (! "lumping")

$$\longrightarrow \quad K_h u_h + B_h p_h = f_h$$

$$B_h^T u_h - \varepsilon M_h p_h = 0$$

$$\left[\begin{array}{c|c} K_h & B_h \\ \hline B_h^T & \varepsilon M_h \end{array}\right] \left\{\begin{array}{c} u_h \\ \hline p_h \end{array}\right\} = \left\{\begin{array}{c} f_h \\ \hline 0 \end{array}\right\}$$

$$\longrightarrow \boxed{p_h = \frac{M_h^{-1}}{\varepsilon} B_h^T u_h}$$

$$\underbrace{\left(K_h + \frac{1}{\varepsilon} B_h M_h^{-1} B_h^T\right)}_{\text{assembled element/element}} u_h = f_h$$