16.346 Astrodynamics
Fall 2008

Lecture 2   The Two Body Problem Continued

The Eccentricity Vector or The Laplace Vector

$$\boxed{\mu\mathbf{e} = \mathbf{v}\times\mathbf{h} - \frac{\mu}{r}\mathbf{r}}$$

Explicit Form of the Velocity Vector **#3.1**

Using the expansion of the triple vector product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$ we have

$$\mathbf{h}\times\mu\mathbf{e} = \mathbf{h}\times(\mathbf{v}\times\mathbf{h}) - \frac{\mu}{r}\mathbf{h}\times\mathbf{r} = h^2\mathbf{v} - (\mathbf{h}\cdot\mathbf{v})\mathbf{h} - \mu\mathbf{h}\times\mathbf{i}_r = h^2\mathbf{v} - \mu h\,\mathbf{i}_h\times\mathbf{i}_r$$

since $\mathbf{h}$ and $\mathbf{v}$ are perpendicular. Therefore:

$$\mathbf{h}\times\mu\mathbf{e} \quad\Longrightarrow\quad \boxed{\mathbf{v} = \frac{\mu}{h}\mathbf{i}_h\times(\mathbf{e}+\mathbf{i}_r)}$$

or

$$\frac{h\mathbf{v}}{\mu} = \mathbf{i}_h\times(e\,\mathbf{i}_e + \mathbf{i}_r) = e\,\mathbf{i}_h\times\mathbf{i}_e + \mathbf{i}_h\times\mathbf{i}_r = e\,\mathbf{i}_p + \mathbf{i}_\theta$$

Then since

$$\mathbf{i}_p = \sin f\,\mathbf{i}_r + \cos f\,\mathbf{i}_\theta$$

we have

$$\boxed{\frac{h\mathbf{v}}{\mu} = e\sin f\,\mathbf{i}_r + (1+e\cos f)\,\mathbf{i}_\theta}$$

which is the basic relation for representing the velocity vector in the Hodograph Plane.

See Page 1 of Lecture 4

Conservation of Energy

$$\frac{h\mathbf{v}}{\mu}\cdot\frac{h\mathbf{v}}{\mu} = \frac{p}{\mu}\mathbf{v}\cdot\mathbf{v} = 2(1+e\cos f) + e^2 - 1 = 2\times\frac{p}{r} - (1-e^2) = p\left(\frac{2}{r} - \frac{1}{a}\right)$$

which can be written in either of two separate forms each having its own name:

Energy Integral $$\boxed{\frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}} = \frac{1}{2}c_3$$

Vis-Viva Integral $$\boxed{v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

The constant c_3 is used by Forest Ray Moulton, a Professor at the University of Chicago in his 1902 book "An Introduction to Celestial Mechanics" — the first book on the subject written by an American.

Conic Sections

Ellipse or Hyperbola in rectangular coordinates $(e \neq 1)$

$$y^2 = r^2 - x^2 = (p - ex)^2 - x^2 = (1 - e^2)[a^2 - (x + ea)^2]$$

$$\boxed{\frac{(x + ea)^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1}$$

Semiminor Axis: $\boxed{b^2 = \left|a^2(1 - e^2)\right| = |a|\, p}$

Ellipse

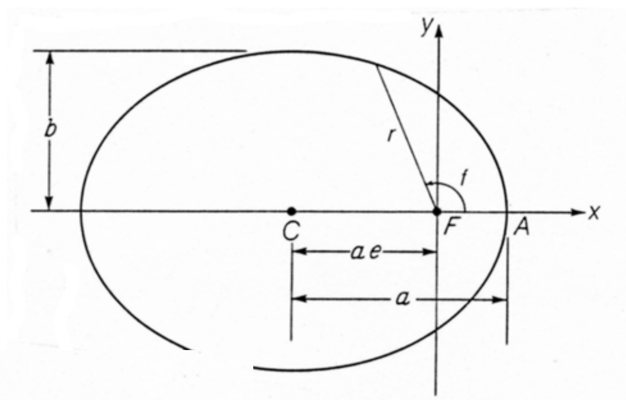
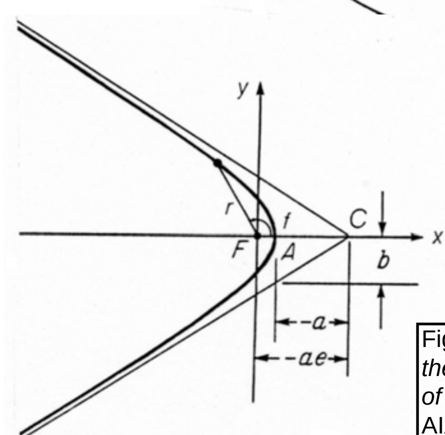


Fig. 3.1 from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.



Hyperbola

Fig. 3.2 from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.

Parabola in rectangular coordinates $(e = 1)$

$$y^2 = r^2 - x^2 = (p - x)^2 - x^2 \quad \Longrightarrow \quad \boxed{y^2 = 2p(\tfrac{1}{2}p - x)}$$

Parabola

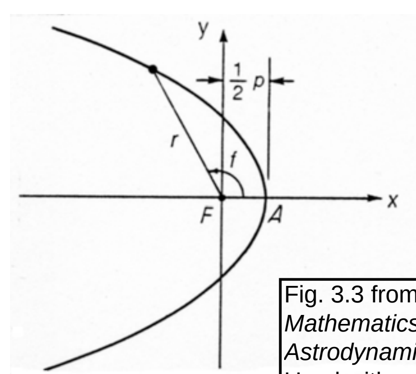


Fig. 3.3 from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.

Alternate Forms of the Equation of Orbit

Origin at focus $r + ex = p$

Origin at center $r + ex = a$

With x now measured from the center which is at a distance ae from the focus, then

$$r + ex = p$$
$$r + e(x - ae) = p = a(1 - e^2)$$
$$r + ex = a$$

Origin at pericenter $r + ex = q$

With x now measured from pericenter which is at a distance of a from the center and a distance of $q = a(1 - e)$ from the focus, then

$$r + ex = p$$
$$r + e(x + q) = p = q(1 + e)$$
$$r + ex = q$$

These are useful to derive other properties of conic sections:

- Focus-Directrix Property: $r = p - ex$: Page 144

$$PF = r = e\left(\frac{p}{e} - x\right) = e \times PN$$

or

$$\boxed{\frac{PF}{PN} = e}$$

- Focal-Radii Property: $r = a - ex$: Page 145

$$PF^2 = (x - ea)^2 + y^2$$
$$PF^{*2} = (x + ea)^2 + y^2$$

so that

$$\begin{aligned} PF^{*2} &= PF^2 + 4aex \\ &= r^2 + 4aex \\ &= (a - ex)^2 + 4aex = (a + ex)^2 \end{aligned}$$

$$PF^* = \begin{cases} a + ex & \text{ellipse} \quad a > 0 \\ -(a + ex) & \text{hyperbola} \quad a < 0, \ x < 0 \end{cases}$$

Thus,

$$\boxed{\begin{array}{ll} PF^* + PF = 2a & \text{ellipse} \\ PF^* - PF = -2a & \text{hyperbola} \end{array}}$$

- Euler's Universal Form: From $r = q - ex$: Page 143

$$y^2 = r^2 - (q + x)^2 = (q - ex)^2 - (q + x)^2$$

Then

$$\boxed{y^2 = -(1 + e)[2qx + (1 - e)x^2]}$$

Basic Two-Body Relations

Vector Equations of Motion $$\frac{d^2\mathbf{r}}{dt^2} + \frac{\mu}{r^3}\mathbf{r} = \mathbf{0} \qquad \text{or} \qquad \frac{d\mathbf{v}}{dt} = -\frac{\mu}{r^3}\mathbf{r}$$

Angular Momentum Vector $$\mathbf{r} \times \frac{d\mathbf{v}}{dt} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{r} \times \mathbf{v} = \textbf{constant} \equiv \mathbf{h}$$

Eccentricity Vector $$\frac{d\mathbf{v}}{dt} \times \mathbf{h} \quad \Longrightarrow \quad \frac{1}{\mu}\mathbf{v} \times \mathbf{h} - \mathbf{i}_r = \textbf{constant} \equiv \mathbf{e}$$

Equation of Orbit $$\mu\mathbf{e} \cdot \mathbf{r} \quad \Longrightarrow \quad r = \frac{h^2/\mu}{1 + e\cos f} = \frac{p}{1 + e\cos f}$$

Velocity Vector $$\mathbf{h} \times \mu\mathbf{e} \quad \Longrightarrow \quad \mathbf{v} = \frac{1}{p}\mathbf{h} \times (\mathbf{e} + \mathbf{i}_r)$$

Orbital Parameter $\boxed{p}$

Dynamics Definition: $p \equiv \dfrac{h^2}{\mu}$ **Geometric Definition:** $p = a(1 - e^2)$

Total Energy or Semimajor Axis or Mean Distance $\boxed{a}$

Dynamics Definition: $$\frac{1}{2}v^2 - \frac{\mu}{r} = \text{constant} \equiv -\frac{\mu}{2a}$$

Geometric Definition: $$\frac{(x + ea)^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

Eqs. of Motion in Polar Coord. $$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 + \frac{\mu}{r^2} = 0 \qquad \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$$

Kepler's Laws

Second Law $$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \text{constant} = \frac{h}{2}$$

First Law $$r = \frac{p}{1 + e\cos f} \qquad \text{or} \qquad r = p - ex$$

Third Law $$\frac{\pi ab}{P} = \frac{h}{2} \quad \Longrightarrow \quad \frac{a^3}{P^2} = \text{constant} = \frac{\mu}{4\pi^2}$$