16.512, Rocket Propulsion
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Lecture 32: Orbital Mechanics: Review, Staging

Mission Planning, Staging

The remaining lectures are devoted to Mission Planning and Vehicle Design, which in reality occurs even before the rocket engines are fully specified (although iterations continuously proceed throughout the process, and engine characteristics do affect the mission plan).

Very roughly, the iteration steps in planning a launch mission are:

(a) Estimate the required ΔV_{TOTAL} using impulsive thrusting formulae, plus add-ons for gravity losses, drag losses, turning losses, etc.

(b) Distribute this ΔV_{TOT} optimally among vehicle stages (since all orbit launches so far require multiple stages in order to avoid carrying empty tankage in the later stages).

(c) Using the mass fractions from (b), perform more detailed flight simulations and refine the partial and total ΔV for the mission.

During stage (b), the total ΔV is assumed to be unchanged when the mass distribution for the stages is varied. This is not strictly true, because often the mission optimization leads to changes in the altitude and velocity at which the various firings are executed and, as we will see, this may alter the various ΔV's. This is the role of stage (c) above.

Another point to be made is that "stages" and "firings" may not map one-to-one. A given stage may be turned off, allowed to coast, and then re-ignited. Or the firing of two consecutive stages may occur with no interruption (or minimal interruption), so that both can be idealized as occurring in the same place. As long as the ΔV's are still regarded as insensitive to mission profile details (as per the comment above), these distinctions do not impact the stage mass calculations, but they can be of great practical importance nonetheless.

Impulsive Thrusting-Gravity Losses. Because of the large accelerations imparted by rocket engines, their firings are usually short, from under one minute to about 10 minutes. In fact, there is a performance incentive in minimizing the firing time, as long as the accelerations remain below structural or other limits. This can be most easily seen in the context of a vertical ascent against gravity. The vehicle's equation of motion is then (ignoring drag)

$$m\frac{dv}{dt} = F - mg \qquad (1)$$

and

$$F = \dot{m}c = -c\frac{dm}{dt} \qquad (2)$$

$$\frac{dv}{dt} = -c \ \frac{d \ln m}{dt} - g \qquad (3)$$

and integrating,

$$\Delta V = V - V_0 = c \ln \frac{m_0}{m} - gt \qquad (4)$$

The "ideal", or gravity-free velocity increment is the familiar $\Delta V_{ideal} = c \ln \frac{m_0}{m}$ (5)

But the presence of gravity reduces the velocity increment by $\Delta V_{Gravity} = gt$ (6)

which can be made insignificant if t is short, but can be very important otherwise. In the limit when the thrust is barely enough to cancel weight, the vehicle just hovers indefinitely with no velocity gain.

In practice, the significant item is the fuel used in the firing, which is contained in the mass ratio m_0/m.

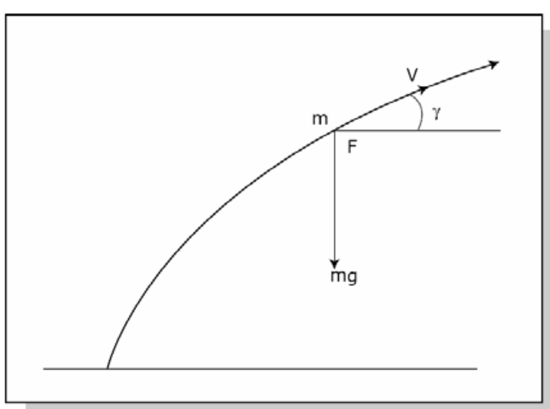
The common procedure is then to first ignore gravity, as if the firing was impulsive (t=0), and calculate the ΔV required for the mission under this assumption. In our simple ascent example, the "mission" is to reach a velocity V, starting at V_0, and so the impulsive ΔV is simply $V-V_0$. From (4) then

$$c \ln \frac{m_o}{m} = \Delta V_{imp.} + gt \qquad (6)$$

and so the extra $\Delta V_{Grav.} = gt$ is added on as a correction, with the implication of additional fuel being used for a given $V-V_0$.

In a more general ascent trajectory (but still over a "flat Earth", since gravity losses occur only near the beginning of flight, before the path becomes nearly horizontal) we would have

$$\frac{dv}{dt} = \frac{F}{m} - g \sin\gamma \qquad (7)$$



$$V \frac{d\gamma}{dt} = -g \cos\gamma \qquad (8)$$

Here we assumed thrust to be aligned with velocity. This is called a gravity turn, and is not the most general maneuver. It is, however, the most economical strategy for turning, since any lateral component of thrust uses propellant without adding flight energy.

Formal integration of (7) now gives

$$\Delta V = c\ \ln\frac{m_0}{m} - \int_0^t g\ \sin\gamma\ dt \tag{9}$$

and so the gravity loss is

$$\Delta V_{Grav.} = \int_0^y g\sin\gamma\, dt \tag{10}$$

Of course, the particular $\gamma(t)$ to be used here must come from simultaneously solving (8) with (7). This solution cannot be done in simple analytical terms when thrust is constant, since a nonlinear 2nd order differential equation is involved. But, interestingly, there is a relatively simple solution when he thrust acceleration $a = \frac{F}{m}$ is assumed constant (i.e., throttling down as mass is consumed). Although this is not a very realistic option, it still is useful in giving information about the initial rotation of the trajectory near the ground, which happens before the mass has time to change much.

Eliminate time by dividing Eqs. (8) and (7) by each other, which separates the variables V and γ :

$$\frac{dV}{V} = -\frac{a - g\sin\gamma}{g\cos\gamma}\ d\gamma \tag{11}$$

We introduce $\frac{a}{g} = n$ and also change angle variable to

$$\Gamma = \tan\left(\frac{\pi}{4} - \frac{\gamma}{2}\right); \qquad \sin\gamma = \frac{1-\Gamma^2}{1+\Gamma^2}; \qquad \cos\gamma = \frac{2\Gamma}{1+\Gamma^2}$$

$$d\gamma = -\frac{2d\Gamma}{1+\Gamma^2} \tag{12}$$

The variable Γ varies between 0 when $\gamma = 90^0$ (initial configuration) to 1 when $\gamma = 0^0$ (orbit insertion). Substituting in (11) and simplifying,

$$\frac{dV}{V} = (n-1)\frac{d\Gamma}{\Gamma} + \frac{2\Gamma\, d\Gamma}{1+\Gamma^2}$$

which can be integrated to

$$\boxed{V = C\,\Gamma^{n-1}\left(1+\Gamma^2\right)} \qquad (13)$$

Here C is a constant of integration. The solution (13) satisfies V = 0 when Γ = *0* (vertical start) for all C (n>1), so C must be calculated by imposing a particular trajectory angle γ (or Γ) at some specified velocity V (or, from later results, at some time or altitude).

The time t is calculated from Eq. (8):

$$dt = -\frac{V\,d\gamma}{g\cos\gamma} = \frac{C}{g}\Gamma^{n-2}\left(1+\Gamma^2\right)d\Gamma$$

or, imposing t = 0 at Γ = *0*.

$$\boxed{t = \frac{C}{g}\left(\frac{\Gamma^{n-1}}{n-1}+\frac{\Gamma^{n+1}}{n+1}\right)} \qquad (14)$$

Similarly, the altitude z follows from $\frac{dz}{dt} = V\sin\gamma$:

$$dz = V\sin\gamma\,dt = \frac{C^2}{g}\left(\Gamma^{2n-3}\;\Gamma^{2n+1}\right)d\Gamma$$

or, with z = 0 at Γ = *0* $z = \frac{C^2}{g}\left(\frac{\Gamma^{2n-2}}{2n-2}-\frac{\Gamma^{2n+2}}{2n+2}\right)$ (15)

We can use this model to calculate gravity losses. Starting from (10), and using the relationships (12),

$$\Delta V_G = \cancel{g}\int_0^{\Gamma}\frac{1-\Gamma^2}{\cancel{1+\Gamma^2}}\frac{C}{\cancel{g}}\Gamma^{n-2}\left(\cancel{1+\Gamma^2}\right)d\Gamma$$

or $\Delta V_G = \frac{C}{g}\left(\frac{\Gamma^{n-1}}{n-1}-\frac{\Gamma^{n+1}}{n+1}\right)$ (16)

We could now use (14) to calculate the constant C by specifying the time to turn to a given angle (Γ). Alternatively, we can eliminate C by division of (16) and (14):

$$\Delta V_G = gt\,\frac{1-\frac{n-1}{n+1}\Gamma_F^2}{1+\frac{n-1}{n+1}\Gamma_F^2} \tag{17}$$

where $\Gamma_F = \Gamma(\gamma_F)$, and γ_F is the angle reached at t, starting from $\gamma = \frac{\pi}{2}$ at t = 0.

As an example, say n = 3, $\gamma_F = 20^0$ (Γ_F = 0.7002). We find from (17)

$$\frac{\Delta V_G}{t} = 5.94\ m/s^2$$

and if t = 60 sec., $\Delta V_G = 357\ m/s$, which is a substantial loss.

An alternative procedure would be to set the velocity V_F reached when $\gamma = \gamma_F$. Eliminating C now between (13) and (16) gives

$$\Delta V_G = V_F\,\frac{\frac{1}{n-1}-\frac{\Gamma^2}{n+1}}{1+\Gamma^2} \tag{18}$$

Say n = 3, V_F = 1,500 m/s, $\gamma_F = 20^0$. We calculate $\Delta V_G = 380\,m/s$ in this case.

<u>Maximum Dynamic Head (“Max-q”) During Ascent</u>

Aerodynamic forces are proportional to $q = \frac{1}{2}\rho V^2$. Initially, $V \simeq 0$ and ρ is high. Later, V increases, but ρ decrease. There is a point of “max-q” in between, which is important for design.

Assume <u>Vertical flight</u> . <u>Neglect drag</u>:

$$m\frac{dv}{dt} = F - mg \qquad \frac{dv}{dt} = \frac{F}{m} - g = (n-1)g \qquad \left(n \equiv \frac{F}{mg}\right)$$

$$\frac{dz}{dt} = v \qquad \text{or} \qquad v\frac{dv}{dz} = (n-1)g$$

Assume n = const. (F ~ m)

$$\frac{v^2}{2} = (n-1)gz$$

Atmospheric "Lapse Rate" "Adiabatic Lapse Rate"

Also, $T = T_0 - \Gamma z$ $\quad \Gamma < \Gamma_a \equiv \dfrac{g}{c_p} = \dfrac{\gamma - 1}{\gamma}\dfrac{g}{R_g} \sim 10\,K/km$

and $dp = -\rho g\,dz = -\dfrac{p}{R_g T}\,g\,dz$ $\quad \dfrac{dp}{p} = -\dfrac{g\,dz}{R_g\left(T_0 - \Gamma z\right)}$

$$\frac{dp}{p} = +\frac{g}{\Gamma R_g}\frac{d\left(T_0 - \Gamma z\right)}{T_0 - \Gamma z} \qquad \frac{p}{p_0} = \left(1 - \frac{\Gamma z}{T_0}\right)^{\frac{g}{\Gamma R_g}}$$

$$\frac{\rho}{\rho_0} = \left(1 - \frac{\Gamma z}{T_0}\right)^{\frac{g}{\Gamma R_g} - 1}$$

$$\boxed{q = \frac{\rho v^2}{2} = \rho_0\left(1 - \frac{\Gamma z}{T_0}\right)^{\frac{g}{\Gamma R_g} - 1}(n-1)\,gz} \tag{19}$$

For q_{MAX} $\quad \dfrac{d\ln q}{dz} = 0$ $\quad \left(\dfrac{g}{\Gamma R_g} - 1\right)\dfrac{\left(-\dfrac{\Gamma}{T_0}\right)}{1 - \dfrac{\Gamma z}{T_0}} + \dfrac{1}{z} = 0$

$$-\frac{g}{R_g T_0} + \frac{\Gamma}{T_0} + \frac{1}{z} - \frac{\Gamma}{T_0} = 0 \qquad \boxed{z_{q_{MAX}} = \frac{R_g T_0}{g}} \tag{20}$$

Some altitude, regardless of acceleration or lapse rate.

Air: $R_g = 287\ J/Kg/K$, $\quad T_0 \simeq 290K$, $\quad g = 9.8\ m/s^2$

$$\boxed{z_{q_{MAX}} = 8,490\,m}$$

Then $\quad q_{MAX} = \rho_0\left(1 - \dfrac{\Gamma}{T_0}\dfrac{R_g T_0}{g}\right)^{\frac{g}{\Gamma R_g} - 1}(n-1)\,g\,\dfrac{R_g T_0}{g}$

$$\boxed{q_{MAX} = \left(1 - \frac{\Gamma R_g}{g}\right)^{\frac{g}{\Gamma R_g} - 1}(n-1)P_0} \tag{21}$$

(proportional to acceleration)

Say $\Gamma = 6\,K/km$ $\qquad \dfrac{\Gamma R_g}{g} = \dfrac{0.006 \times 287}{9.8} = 0.176$

and n = 3

$$q_{MAX} = \left(1 - 0.176\right)^{\frac{1}{0.176}-1}\left(3-1\right)P_0 = 0.808 \text{ atm}$$

1 atm

$$= 0.808 \times \left(14.7 \times 12^2\right) = 1710\,\text{psf}$$

Also, then $v^2_{Max\,q} = 2\left(n-1\right)\not{g}\dfrac{R_g T_0}{\not{g}}$ $\qquad \boxed{M'^{\,2}_{Max_q} = \dfrac{2}{\gamma}\left(n-1\right)}$ $\qquad$ (based on C_0, at ground)

Based on local T, $M^2_{Max\,q} = \dfrac{2}{\gamma}\left(n-1\right)\dfrac{T_0}{T} = \dfrac{2\left(n-1\right)}{\gamma}\dfrac{1}{1 - \dfrac{\Gamma}{\not{T_0}}\dfrac{R_g \not{T_0}}{g}}$

$$\boxed{M'^{\,2}_{Max_q} = \frac{2\left(n-1\right)}{\gamma\left(1 - \dfrac{\Gamma R_g}{g}\right)}} \qquad M^2 = \frac{2 \times 2}{1.4\left(1-0.176\right)} \qquad \boxed{M_{Max\,q} = 1.862}$$

Drag Losses: Like gravity losses, drag losses are important only near the ground, peaking somewhat above $z\left(q_{MAX}\right)$. Therefore, they should be estimated and added to the 1st stage ΔV budget alone. The “drag loss” is defined by analogy to ΔV_G as the decrease in velocity due to the accumulated drag deceleration:

$$\Delta V_D = \int_0^{\text{"}\infty\text{"}} \frac{D}{m}\,dt \qquad (22)$$

Drag is D = q C_D A, where A is the frontal area, and C_D varies with vehicle shape and Mach number (from about 0.02 at low M to a peak of perhaps 0.15 in transonic flow, then decreasing again). For estimation purposes only, we will use a mean $C_D = \overline{C_D}$ and write (22) as

$$\Delta V_D = \frac{A\,\overline{C}_D}{M_0}\int q\,\frac{m_0}{m}\,\frac{dz}{v} \qquad (23)$$

Our estimate will be based on quantities evaluated at q_{MAX}, and an effective $\Delta z \sim 3\,z\left(q_{MAX}\right)$:

$$\Delta V_D \simeq \frac{A\,\bar{C}_D}{m_0} q_{MAX} \left(\frac{m_0}{m}\right)_{q_{MAX}} \frac{3z\left(q_{MAX}\right)}{v\left(q_{MAX}\right)} \qquad (24)$$

The "ballistic coefficient" $\frac{A\,\bar{C}_D}{M_0}$ can be related to the vehicle length L and its mean density $\bar{\rho}$. Assuming an given shape with (Volume) = $\frac{2}{3}$ AL, we find

$$\frac{A\,\bar{C}_D}{M_0} = \frac{3}{2}\,\frac{\bar{C}_D}{\bar{\rho}L} \qquad (25)$$

The mass ratio $\frac{m_0}{m} = e^{-\frac{v}{c}}$ can be estimated using $v = \sqrt{2\left(n-1\right)g}$ and so

$$\left(\frac{m_0}{m}\right)_{q_{MAX}} = e^{-\frac{\sqrt{2(n-1)R_g T_0}}{c}} \qquad (26)$$

Using as well the values found previously for q_{MAX} and $z\left(q_{MAX}\right)$, and simplifying, our approximate expression is

$$\Delta V_D \simeq 4.5\;\bar{C}_D\sqrt{\frac{n-1}{2}R_g T_0}\;\frac{P_0}{\bar{\rho}\,gL}\left(1-\frac{\Gamma R_g}{g}\right)^{\frac{g}{\Gamma R_g}-1} e^{-\frac{\sqrt{2(n-1)R_g T_0}}{c}} \qquad (27)$$

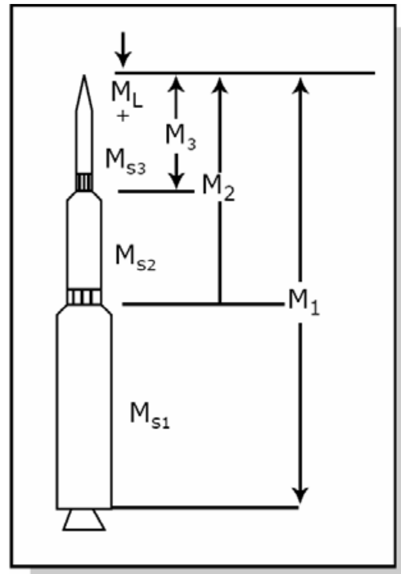
For an example, take $\bar{C}_D = 0.1$, n = 3, T_0 = 290 K, $\bar{\rho}$ = 500 Kg/m^3 (half the water density), Γ = 6 K/km = 0.006 K/m, and c = 3,000 m/s. We calculate

$$\Delta V_D \simeq \frac{1060}{L\left(m\right)}\left(m/s\right) \qquad (28)$$

For a large vehicle (say, L = 30 m) this is small ($\Delta V_D = 35$ m/s). But for a 3 m. vehicle this amounts to $\Delta V_D = 353$ m/s, a substantial loss. The difference can be traced to the larger Area/Volume of the smaller vehicle.

To conclude, note the dependence $\Delta V_D \sim \sqrt{n-1}$, which shows that fast-accelerating vehicles, like interception missiles, suffer more drag losses than slowly accelerating ones. There is here a tradeoff with gravity losses, which vary in the opposite manner.

Optimum Staging



$$M_{si} = \varepsilon_i M_i \qquad M_{i+1} = M_{L_i} = \lambda_i M_i$$

$$\underbrace{M_{L_i} + M_{S_i}}_{M_{i+1}} = M_{f_i} = e^{-\frac{\Delta V_i}{L_i}} M_i$$

$$\frac{M_{i+1}}{M_i} = e^{-\frac{\Delta V_i}{C_i}} - \varepsilon_i \qquad \frac{M_L}{M_{n-1}} \frac{M_{n-1}}{M_{n-2}} \cdots \frac{M_2}{M_1} = \frac{M_L}{M_1}$$

$$\frac{M_L}{M_1} = \mathop{\pi}_{i=1}^{n} \left(e^{-\frac{\Delta V_i}{C_i}} - \varepsilon_i \right)$$

Maximize subject to $\sum_i \Delta V_i = \Delta V$ (assume ε_i is independent of M_i. In reality it may depend on absolute mass.)

$$\phi = \ln \frac{M_L}{M_0} - \alpha \left(\sum_i V_i \right) = \sum_i \left[\ln \left(e^{-\frac{\Delta V_i}{C_i}} - \varepsilon_i \right) - \alpha \, \Delta V_i \right]$$

For each i, $\dfrac{\partial \phi}{\partial \Delta V_i} = \dfrac{-\frac{1}{c_i} e^{-\frac{\Delta V_i}{c_i}}}{e^{-\frac{\Delta V_i}{c_i}} - \varepsilon_i} - \alpha \qquad \dfrac{1}{\alpha} = -c_i \left(1 - \varepsilon_i \, e^{\frac{\Delta V_i}{c_i}} \right)$

$$\frac{1}{\alpha c_i} + 1 = \varepsilon_i e^{\frac{\Delta V_i}{c_i}} \qquad \boxed{\Delta V_i = C_i \ln \left(\frac{1 + \frac{1}{\alpha \, c_i}}{\varepsilon_i} \right)}$$

Then, find α from $\boxed{\sum_{i=1}^{n} c_i \ln \left(\frac{1 + \frac{1}{\alpha c_i}}{\varepsilon_i} \right) = \Delta V}$, then find ΔV_i from

Assuming $c_i = c$ (same all stages), then

$$\frac{\Delta V}{c} = \sum_{i=1}^{n} \ln\left(\frac{1+\frac{1}{\alpha c}}{\varepsilon_i}\right) = \ln\left[\frac{\left(1+\frac{1}{\alpha c}\right)^n}{\pi \varepsilon_i}\right]$$

$$\left(1+\frac{1}{\alpha c}\right)^n = \left(\underset{i}{\pi}\,\varepsilon_i\right)^{\frac{1}{n}} e^{\frac{\Delta V}{nc}} \qquad \alpha = \frac{-\frac{1}{c}}{1-<\varepsilon_i>_G} e^{+\frac{\Delta V}{nc}} \qquad <\varepsilon>_G = \left(\underset{i}{\pi}\,\varepsilon_i\right)^{\frac{1}{n}}$$

$$\Delta V_i = c\left[\ln\left(1+\frac{1}{\alpha c}\right) - \ln \varepsilon_i\right] = c\left[\ln <\varepsilon> + \frac{\Delta V}{nc} - \ln \varepsilon_i\right] \qquad \boxed{\frac{\Delta V_i}{c} = \frac{\Delta V}{nc} - \ln \frac{\varepsilon_i}{<\varepsilon>}}$$

So, $$\left(\frac{M_L}{M_I}\right)_{OPT} = \underset{i=1}{\overset{n}{\pi}}\left[e^{-\left(\frac{\Delta V}{nc} - \ln\frac{\varepsilon_i}{<\varepsilon>}\right)} - \varepsilon_i\right] = \underset{i=1}{\overset{n}{\pi}}\left(\frac{\varepsilon_i}{<\varepsilon>} e^{-\frac{\Delta V}{nc}} - \varepsilon_i\right) = \left(\underset{i=1}{\overset{n}{\pi}}\varepsilon_i\right)\left(\frac{e^{-\frac{\Delta V}{nc}}}{<\varepsilon>} - 1\right)^n$$

So, less ΔV_i when stage is less structurally efficient.

$$\boxed{\left(\frac{M_L}{M_I}\right)_{OPT} = \left(e^{-\frac{\Delta V}{nc}} - <\varepsilon>\right)^n}$$

Note:

Meaning of $\boxed{\alpha = \frac{\partial\left(\ln \frac{M_L}{M_0}\right)}{\partial \Delta V} < 0}$

Sensitivity of payload ratio to overall ΔV changes (after re-optimizing)

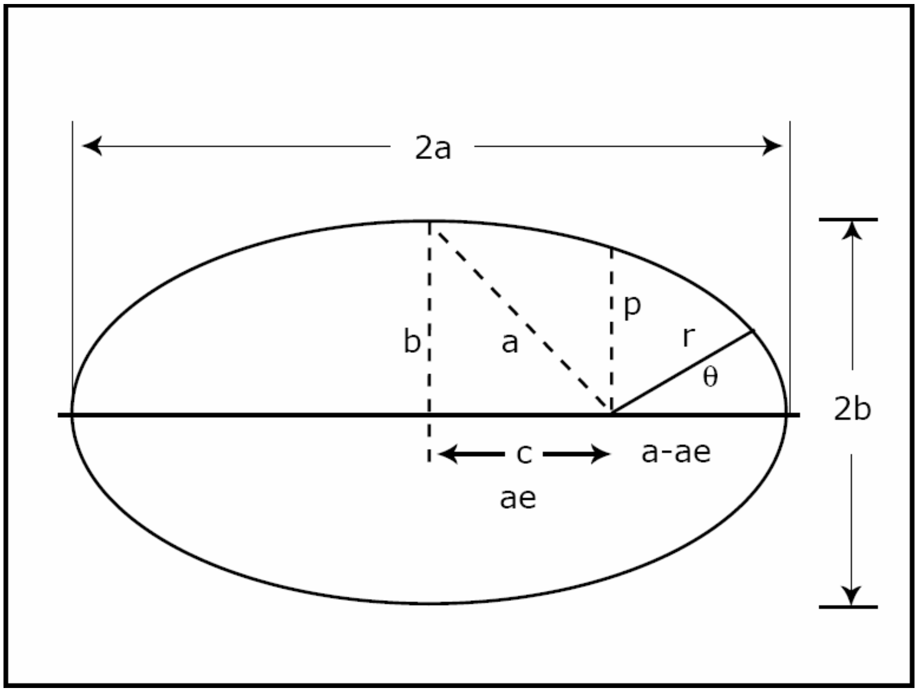
Generally: Max $\underset{i=1 \text{ to } n}{f(x_i)}$ given $\underset{j=1 \text{ to } m<n}{g_j(x_i) = G_j}$ $\qquad \phi = f - \sum_j \lambda_j g_j$

$$dG_j = \sum_i \frac{\partial G_j}{\partial x_i} dx_i = \sum_i \frac{\partial g_i}{\partial x_i} dx_i \qquad \text{and} \qquad \frac{\partial f}{\partial x_i} = \sum_j \lambda_j \frac{\partial g_j}{\partial x_i}$$

$$\partial f = \sum_i \frac{\partial f}{\partial x_i} dx_i = \sum_i \left(\sum_j \lambda_j \frac{\partial g_i}{\partial x_i}\right) dx_i = \sum_j \lambda_j \sum_i \frac{\partial g_i}{\partial x_i} dx_i = \sum_j \lambda_j dG_j$$

So, $\boxed{\lambda_j = \left(\frac{\partial f}{\partial G_j}\right)_{\text{at optimum}}}$

Review of Orbital Dynamics (Single center)



$$r = \frac{p}{1 + e\cos\theta}$$

(θ from perigee)

↑ "true anomaly"

$$e = \frac{c}{a} = \sqrt{1 - \left(\frac{b}{a}\right)^2} \quad ;$$

Apoapse (apogee, aphelion): $\theta = \pi \rightarrow r_a = \dfrac{p}{1-e}$

Periapse (perigee, perihelion): $\theta = 0 \rightarrow r_p = \dfrac{p}{1+e}$

$$r_a + r_p = p\left(\frac{1}{1-e} + \frac{1}{1+e}\right) = 2a$$

$$p\,\frac{2}{1-e^2} = 2a$$

$$\rightarrow \boxed{p = a\left(1 - e^2\right)}$$

$$\rightarrow \boxed{r_p = a(1-e), r_a = a(1+e)}$$

Energy Conservation: $\frac{1}{2}v^2 - \frac{\mu}{r} = E$ $\qquad (\mu = GM)$

At perigee $\quad \frac{1}{2}v_p^2 - \frac{\mu}{a(1-e)} = E$

At apogee $\quad \frac{1}{2}v_a^2 - \frac{\mu}{a(1+e)} = E$

$$\left(\frac{v_p}{v_a}\right)^2 = \frac{E + \frac{\mu}{a(1-e)}}{E + \frac{\mu}{a(1+e)}} \qquad *$$

Angular momentum conservation: $r\,v_\theta = h$ (or $r^2\,\dot\theta = h$)

at perigee: $\quad h = a(1-e)\,v_p$

at apogee: $\quad h = a(1+e)\,v_a$

$$\boxed{\frac{v_p}{v_a} = \frac{1+e}{1-e}} \qquad **$$

equate (*) = (**) $\quad \left(\frac{1+e}{1-e}\right)^2 = \frac{E + \frac{\mu}{a(1-e)}}{E + \frac{\mu}{a(1+e)}} \qquad \left(\frac{1+e}{1-e}\right)^2 E + \frac{\mu}{a}\frac{1+e}{(1-e)^2} = E + \frac{\mu}{a(1-e)}$

$$E\left[\frac{(1+e)^2}{(1-e)^2} - 1\right] = \frac{\mu}{a(1-e)}\left(1 - \frac{1+e}{1-e}\right)$$

$$E\,\frac{4e}{(1-e)^2} = \frac{\mu}{a(1-e)}\,\frac{-2e}{(1-e)}$$

$\boxed{E = -\frac{\mu}{2a}}$ ← indep. of e (given a)

and then $\quad v_p^2 = \frac{2\mu}{a(1-e)} - \frac{\mu}{a} = \frac{\mu}{a}\,\frac{1+e}{1-e}$

$$v_p = \sqrt{\frac{\mu}{a}\,\frac{1+e}{1-e}} = \sqrt{\frac{\mu}{r_p}\,\frac{2r_a}{r_a + r_p}}$$

and

$$v_a = \sqrt{\frac{\mu}{a}\,\frac{1-e}{1+e}} = \sqrt{\frac{\mu}{r_p}\,\frac{2r_p}{r_a + r_p}}$$

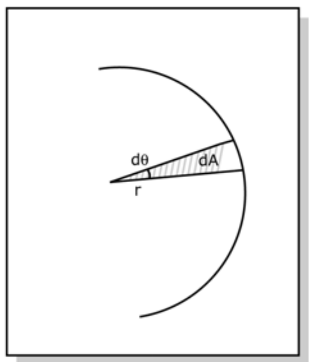
and $\quad \left[h = a(1-e)\sqrt{\frac{\mu}{a}\,\frac{1+e}{1-e}} = \sqrt{\mu\,a(1-e^2)}\right]$ $\quad$ or $\quad \boxed{h = \sqrt{\mu\,p}}$

Period

$$r^2\dot{\theta} = h$$

$$dA = \frac{1}{2} r\left(r\,d\theta\right) \rightarrow \frac{dA}{dt} = \frac{h}{2} \qquad A = \frac{h}{2}T \qquad \boxed{T = \frac{2A}{h}}$$

$$\left.\begin{array}{l} A = \pi ab = \pi a^2\sqrt{1-e^2} \\ h = \sqrt{\mu a\left(1-e^2\right)} \end{array}\right\} \quad \boxed{T = 2\pi\frac{a^{3/2}}{\sqrt{\mu}}} \leftarrow \text{indep. of e}$$



Velocity: From energy conservation $\frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}$

$$\boxed{v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}}$$

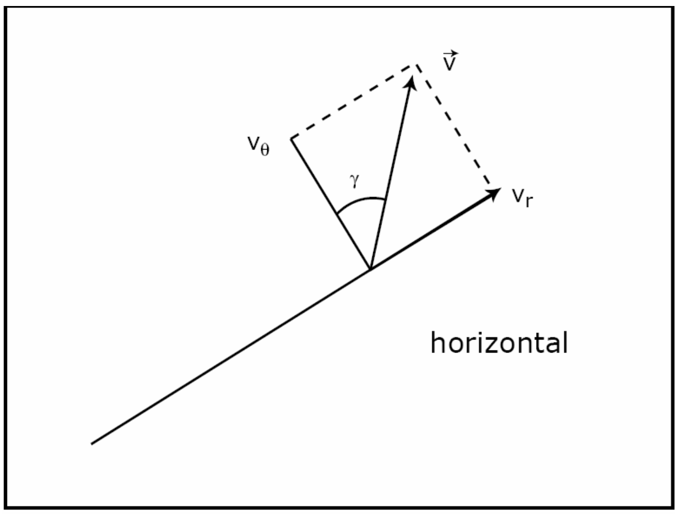
$$v_\theta = \frac{h}{r} \qquad \boxed{\begin{array}{l} v_\theta = \dfrac{\sqrt{\mu a\left(1-e^2\right)}}{r} = \dfrac{\sqrt{\mu r_a r_p \,/\left(r_a + r_p\right)}}{r} \\ \quad = r\,\dot{\theta} \end{array}} \qquad \boxed{v_r = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a} - \frac{\mu a\left(1-e^2\right)}{r^2}} = \dot{r}}$$

Time in orbit:

$$\frac{d\theta}{dt} = \frac{h}{r^2} = \frac{\sqrt{\mu a\left(1-e^2\right)}}{a^2\left(1-e^2\right)}\left(1 + e\cos\theta\right)^2 \qquad \rightarrow \qquad t = t\left(\theta\right)$$

not easy – Lambert's prob.
(except for full orbit)

Path angle:



$$\tan\gamma = \frac{v_r}{v_\theta} = \frac{dr}{r\,d\theta} = +\frac{e(+\sin\theta)}{1+e\cos\theta}$$

Circular orbits $\quad r = a \quad \rightarrow \quad \boxed{v = \sqrt{\frac{\mu}{r}}} \qquad \left[T = \frac{2\pi v}{v} = 2\pi\sqrt{\frac{r^3}{\mu}}\right]$ check,

Time in orbit (elliptic case)

$$\frac{d\theta}{dt} = \sqrt{\frac{\mu}{a^3\left(1-e^2\right)^3}}\left(1+e\cos\theta\right)^2$$

$$\frac{1+\cos\theta}{2} = \cos^2\frac{\theta}{2} = \frac{1}{1+t^2}$$

$$\sqrt{\frac{\mu}{a^3\left(1-e^2\right)^3}}\,dt = \frac{d\theta}{\left(1+e\cos\theta\right)^2} \qquad \tan\frac{\theta}{2} = t \qquad \cos\theta = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2} \qquad d\theta = \frac{2dt}{1+t^2}$$

$$= \frac{2\left(1+t^2\right)dt}{\left(1+t^2+e-et^2\right)^2} = \frac{2}{\left(1+e\right)^2}\;\frac{1+t^2}{\left(1+\frac{1-e}{1+e}t^2\right)^2}dt$$

Define E by $\frac{1-e}{1+e}t^2 = \tan^2\frac{E}{2}$ $\qquad t = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$ $\qquad dt = \sqrt{\frac{1+e}{1-e}}\,\frac{\frac{1}{2}dE}{\cos^2\frac{E}{2}}$

$$\sqrt{\frac{\mu}{a^3\left(1-e^2\right)^3}}\,dt = \frac{\cancel{2}}{(1+e)^2}\,\frac{1+\frac{1+e}{1-e}\tan^2\frac{E}{2}}{\left(1+\tan^2\frac{E}{2}\right)^{\cancel{2}}}\sqrt{\frac{1+e}{1-e}}\,\frac{dE/\cancel{2}}{\cancel{\cos^2\frac{E}{2}}}$$

$$\sqrt{\frac{\mu}{a^3}}\,dt = \frac{1-e^2}{1+e}\,\frac{1+\frac{1+e}{1-e}\tan^2\frac{E}{2}}{\left(1+\tan^2\frac{E}{2}\right)}dE = \frac{1-e^2}{1+e}\left(\cos^2\frac{E}{2}+\frac{1+e}{1-e}\sin^2\frac{E}{2}\right)dE = \frac{1-e^2}{1+e}\left(\frac{1+\cos E}{2}+\frac{1+e}{1-e}\,\frac{1-\cos E}{2}\right)dE$$

$$\sqrt{\frac{\mu}{a^3}}\,dt = \frac{1-e^2}{1+e}\left(\frac{1}{1-e}-\frac{e}{1-e}\,\cos E\right)dE = \frac{(1-e\cos E)}{\cancel{1-e^2}}\,dE$$

$$\boxed{\sqrt{\frac{\mu}{a^3}}\,dt = E - e\sin E}$$ (t from perigee passage)

with $$\boxed{E = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right)}$$

from which $$\cos E = \frac{1-\tan^2\frac{E}{2}}{1+\tan^2\frac{E}{2}} = \frac{1-\frac{1-e}{1+e}\tan^2\frac{\theta}{2}}{1+\frac{1-e}{1+e}\tan^2\frac{\theta}{2}} = \frac{1+e-(1-e)\tan^2\frac{\theta}{2}}{1+e+(1-e)\tan^2\frac{\theta}{2}}$$

$$t^2\cos\theta + \cos\theta = 1 - t^2 \qquad t^2 = \frac{1-\cos\theta}{1+\cos\theta} \qquad \cos E = \frac{1+\cos\theta - \frac{1-e}{1+e}(1-\cos\theta)}{1+\cos\theta+\frac{1-e}{1+e}(1-\cos\theta)}$$

$$\cos E = \frac{2e+2\cos\theta}{2+2e\cos\theta} \qquad \boxed{\cos E = \frac{e+\cos\theta}{1+e\cos\theta}} \quad\Rightarrow\quad \cos\theta = \frac{\cos E - e}{1-e\cos E} \qquad 1+e\cos\theta = \frac{1-e^2}{1-e\cos E}\ *$$

So, directly

$$\sqrt{\frac{\mu}{a^3}}\,dt = \frac{(1-e^2)^{3/2}}{(1-e^2)^2}\,d\theta\,(1-e\cos E)^2 \qquad \Big\| \qquad \sin\theta = \frac{\sqrt{(1-e\cos E)^2-(\cos E-e)^2}}{1-e\cos E} = \frac{\sqrt{1-e^2}\,\sin E}{1-e\cos E}$$

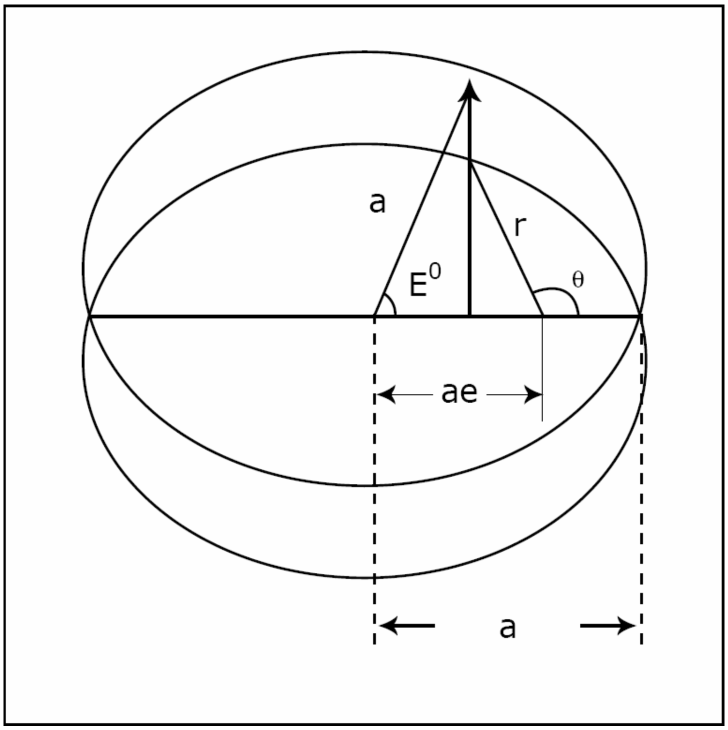
$$\sqrt{\frac{\mu}{a^3}}\,dt = \frac{1}{\sqrt{1-e^2}}(1-e\cos E)\frac{1-e^2}{\sqrt{1-e^2}}\,dE = (1-e\cos E)\,dE$$

$$\frac{\sqrt{1-e^2}\;\cancel{\sin E}}{\cancel{1-e\cos E}}\,d\theta = \frac{\cancel{\sin E}\,(1-\cancel{e\cos E}) + (\cancel{\cos E} - e\sin E)}{(1-e\cos E)^{\cancel{2}}}$$

$$\boxed{\sqrt{\frac{\mu}{a^3}}\;t = E - e\sin E}$$

$$\sqrt{1-e^2}\,d\theta = \frac{1-e^2\,dE}{1-e\cos E}$$

From (**) $\quad r = \dfrac{P}{1+e\cos\theta} = \dfrac{a\,\cancel{(1-e^2)}}{\cancel{(1-e^2)}}(1-e\cos\theta) \qquad r = a(1-e\cos E)$



$$ae - a\cos E = r(-\cos\theta)$$

$$\cancel{a}\,(\cos E - e) = \cancel{a}\,(1-e\cos E)\cos\theta$$

$$\cos\theta = \frac{\cos E - e}{1-e\cos E}$$