16.522: Session 18
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HALL THRUSTERS: FLUID MODEL OF THE DISCHARGE

Types of models

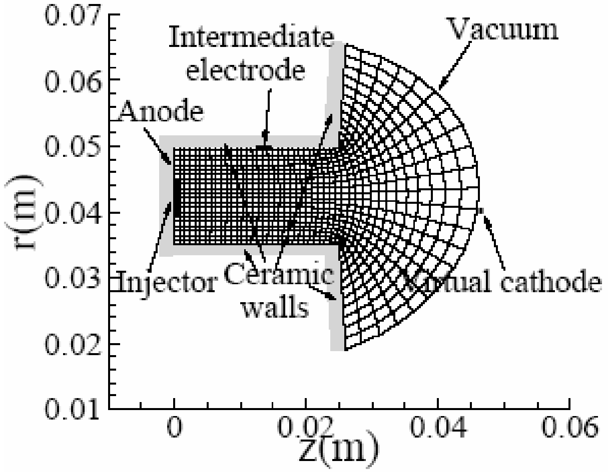
- **Kinetic:** based on Boltzmann eq.; unaffordable except for particular aspects of the problem
- **Fluid:** familiar formulation but important difficulties arising from
  1. Weak collisionality (Kn large)
  2. Wall interaction
  3. Curved magnetic topology
  4. 2D subsonic/supersonic ion flows
- **Particle In Cell & MonteCarlo methods (PIC/MCC):** good for weak collisionality; simple to implement, but subject to ‘numerical effects’; important difficulties in dealing with disparate scales of electron and ions. MCC model collisions statistically.
- **Hybrid (PIC/MCC for heavy species & fluid for electrons):** best compromise today; allows 2D (geometrical and magnetic) effects; avoids small electron scales and admits quasineutrality

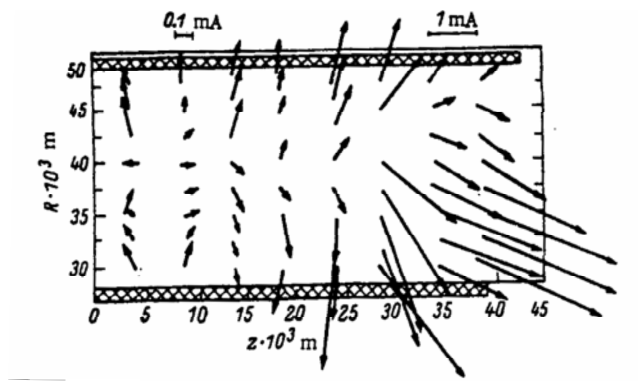
Simulation of the plasma discharge

- 2D, axisymmetric model
- Quasineutral plasma except for sheaths around walls.
- Plasma wall interaction treated in separate sheath models.
- Boundaries: 1) anode + gas injector, 2) cathode surface, 3) lateral walls
- 3 or 4 species: neutrals, ions (+, ++), and electrons
- Ion dynamics: unmagnetized, near collisionless, internal regular sonic transitions, singular sonic transitions at sheath edges.
- Electrons: magnetized, diffusive motion, and weakly collisional (local thermodynamic equilibrium is not assured)
- Fluid modelling: complex and uncertain.
- There is no fully 2D model yet. Two existing options:

  a) Approximate 1D (axial) fluid model

  b) Near 2D hybrid model: fluid eqs. for electrons; particle model for ions & neutrals





2D fluid model

- Basic hypotheses: 1) azimuthal symmetry: $\dfrac{\partial}{\partial \theta} = 0,\ u_{\theta i} = 0,\ u_{\theta n} = 0$

  2) Quasineutrality: $\boxed{n_e = n_{i+} + 2n_{i++}}$ $\Rightarrow$ boundary conditions at sheath edges

  3) Simplified treatment of pressure tensors

- Continuity equations: $\boxed{\dfrac{\partial n_e}{\partial t} + \nabla \cdot n_e \vec{u}_e = S_{ion}}$ $\qquad S_{ion} \approx n_e n_n R_{ion}(T_e) \leftarrow$ ionization

$$\boxed{\frac{\partial n_i}{\partial t} + \nabla \cdot n_i \vec{u}_i = S_{ion}, \qquad \frac{\partial n_n}{\partial t} + \nabla \cdot n_n \vec{u}_n = -S_{ion}}$$

(here $n_e = n_i$ only)

- Electron momentum: $\boxed{\cancel{\dfrac{\partial}{\partial t}(m_e n_e \vec{u}_e)} + \cancel{\nabla \cdot m_e n_e \vec{u}_e \vec{u}_e} = -\nabla(n_e T_e) - e n_e(\vec{E} + \vec{u}_e \times \vec{B}) + \vec{M}_e}$

$$\vec{M}_e \approx -n_e n_n R_{en}(T_e) m_e \vec{u}_e \qquad \leftarrow \qquad \text{e-n momentum transfer}$$

- Ion momentum: $\boxed{\dfrac{\partial}{\partial t}(m_i n_i \vec{u}_i) + \nabla \cdot m_i n_i \vec{u}_i \vec{u}_i = \cancel{-\nabla(n_i T_i)} + e n_i \vec{E} + \vec{M}_i}$

$$\vec{M}_i \approx S_{ion} m_i \vec{u}_n - S_{cx} m_i (\vec{u}_i - \vec{u}_n) \qquad \leftarrow \qquad \text{ionization + charge-exchange mom. transf.}$$

2D fluid model

- Electron (total) energy equation :

$$\frac{\partial}{\partial t}\left(\frac{3}{2}T_e + \frac{1}{2}m_e u_e^2\right)n_e + \nabla\cdot\left[\left(\frac{5}{2}T_e + \frac{1}{2}m_e u_e^2\right)n_e\vec{u}_e + \vec{q}_e\right] = -en_e\vec{u}_e\cdot\vec{E} + Q_e$$

$$Q_e \approx -n_e n_n R_{ion}(T_e)E_{ion}\alpha_{ion} \leftarrow \text{ ionization } \quad (E_{ion} = 12.1\text{eV}, \quad \alpha_{ion} \sim 1.5-2.5 \text{ for Xe})$$

- Internal energy = total energy – mechanical energy

$$\frac{\partial}{\partial t}\left(\frac{3}{2}T_e n_e\right) + \nabla\cdot\left[\frac{3}{2}T_e n_e\vec{u}_e + \vec{q}_e\right] = -n_e T_e\nabla\cdot\vec{u}_e + Q'_e$$

$$Q'_e \equiv Q_e - \vec{M}_e\cdot\vec{u}_e + \frac{1}{2}m_e u_e{}^2 S_e \approx n_e n_n\left[\left(R_{en} + \frac{1}{2}R_{ion}\right)m_e u_e{}^2 - R_{ion}E_{ion}\alpha_{ion}\right]$$

- Heat flux (diffusive model): (Bittencourt)

$$\vec{o} = -\frac{5}{2}p_e\nabla T_e - e\vec{q}_e\wedge\vec{B} - m_e\nu_e\vec{q}_e \quad \rightarrow \quad \vec{q}_e = -\bar{\bar{K}}_e\cdot\nabla T_e$$

- Energy equations for ions and neutrals.

From 2D to 1D model

- Electron continuity equation: $$\frac{\partial n_e}{\partial t}+\frac{\partial n_e u_{ze}}{\partial z}+\frac{1}{r}\frac{\partial r n_e u_{re}}{\partial r}=n_e \nu_i$$
- The <u>1D axial model</u> works with values averaged over each ('doughnut') radial section

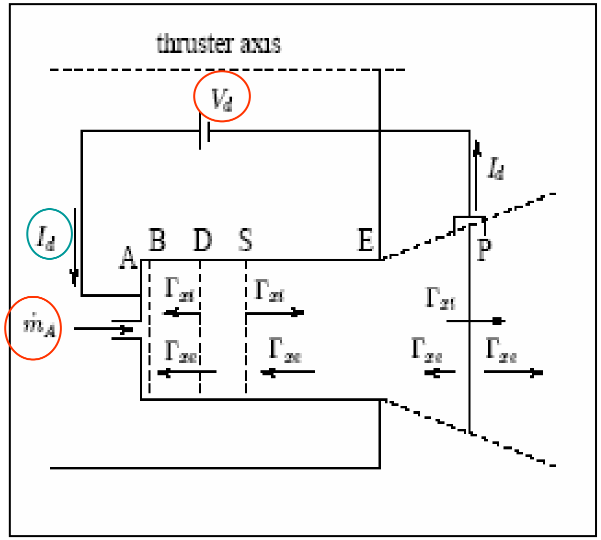
$$0=\frac{2}{r_{ext}^2-r_{\mathrm{int}}^2}\int_{r_{\mathrm{int}}}^{r_{ext}}\left[\frac{\partial n_e}{\partial t}+\frac{\partial n_e u_{ze}}{\partial z}+\frac{1}{r}\frac{\partial r n_e u_{re}}{\partial r}-n_e \nu_i\right] \rightarrow \frac{\partial \bar{n}_e}{\partial t}+\frac{\partial \bar{n}_e \bar{u}_{ze}}{\partial z}=\bar{n}_e(\bar{\nu}_i-\bar{\nu}_w)$$

with $\bar{n}_e \equiv \bar{n}_e(z),...,$ and 'source' $\bar{n}_e\bar{\nu}_w=2\frac{(r n_e u_{re})|_{\mathrm{int}}^{ext}}{r_{ext}^2-r_{\mathrm{int}}^2}$ evaluating losses at lateral walls.

- An auxiliary <u>radial model</u> (at each z) is needed to

  compute $\bar{\nu}_w$ & determine radial profiles: $\frac{1}{r}\frac{\partial r n_e u_{re}}{\partial r}\approx n_e \bar{\nu}_w$

- This is a variable-separation type of solution; $\bar{\nu}_w(z)$ is an eigenvalue of the radial model.
- We proceed similarly with the rest of fluid equations.



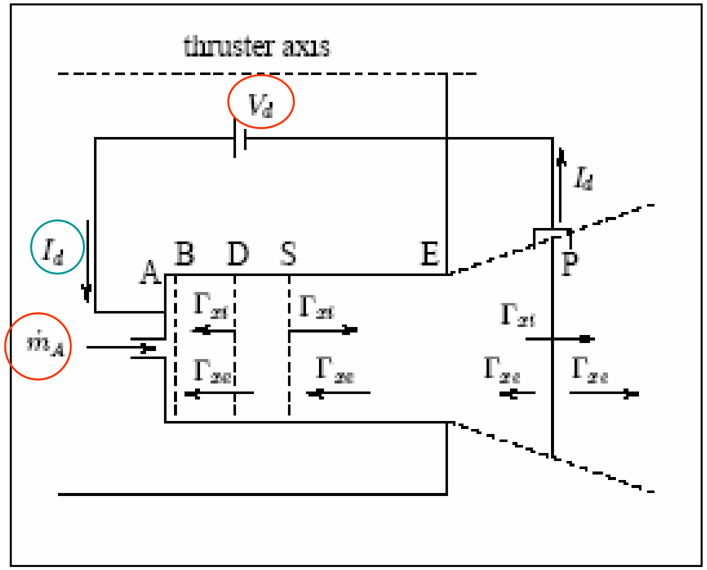
1D axial model

- Neglect jet divergence, doubly-charged ions, $p_i, p_n, \ldots$
- Wall interaction terms appear as source terms (instead of BCs)
- Conservation of species flows: $$\frac{d}{dz}(n_e u_{zi}) = n_e(\nu_i - \nu_w)$$

ν_i : ionization frequency $\quad \nu_w$: recombination frequency

$$n_n u_{zn} + n_e u_{zi} = \text{const} = \dot{m}_A / A m_i$$

$$n_e u_{zi} - n_e u_{ze} = \text{const} = I_d / Ae$$



- Ion axial momentum equation:

$$\frac{d}{dz}(m_i n_e u_{zi}{}^2) = -e n_e \frac{d\phi}{dz} + m_i n_e(\nu_i u_{zn} - \nu_w u_{zi})$$

- Neutral axial momentum equation

$$\frac{d}{dz}(m_i n_n u_{zn}{}^2) = m_i n_e(\nu_{wn} u_{znw} - \nu_i u_{zn})$$

u_{znw} velocity of neutrals from recombination $(\neq u_{zi})$

1D axial model

- Electron Ohm´s law: $$-u_{ze}=\frac{\nu_e}{m_e\omega_e^2}\left[e\frac{d\phi}{dz}-\frac{1}{n_e}\frac{d}{dz}(n_eT_e)\right]$$ $$u_{\theta e}=-u_{ze}\frac{\omega_e}{\nu_e}$$

$\nu_e=\nu_{en}+\nu_{ei}+\nu_{wm}+\nu_{turb}$: effective collision frequency

(ν_{wm} due to wall interaction, ν_{turb} due to plasma turbulence)

- Electron internal energy:

$$\frac{d}{dz}\left(\frac{5}{2}n_eT_eu_{ze}+q_{ze}\right)=u_{ze}\frac{d(n_eT_e)}{dz}+\nu_en_em_eu_e^2-\nu_in_eE_{ion}\alpha_{ion}-\beta_e\nu_wn_eT_e$$

($\beta_e\sim 6-100$: factor for energy losses at lateral walls)

- Heat conduction: $0=-\frac{5}{2}p_e\nabla T_e-e\vec{q}_e\wedge\vec{B}-m_e\nu_e\vec{q}_e \quad\rightarrow\quad q_{ze}=-\frac{5n_eT_e}{2m_e}\frac{\nu_e}{\omega_e^2}\frac{dT_e}{dz}$

- Normalized set of equations: $(T_e-m_iu_{zi}^2)\frac{d\vec{Y}}{dz}=\vec{F}(\vec{Y})$

$\vec{Y}=(n_e,n_n,u_{zi},u_{ze},u_{zn},T_e,q_{ze},\phi)$ vector of 8 plasma variables

- Singular (sonic) points of the equations at : $M\equiv\frac{u_{zi}}{\sqrt{T_e/m_i}}=\pm 1$

1D axial model

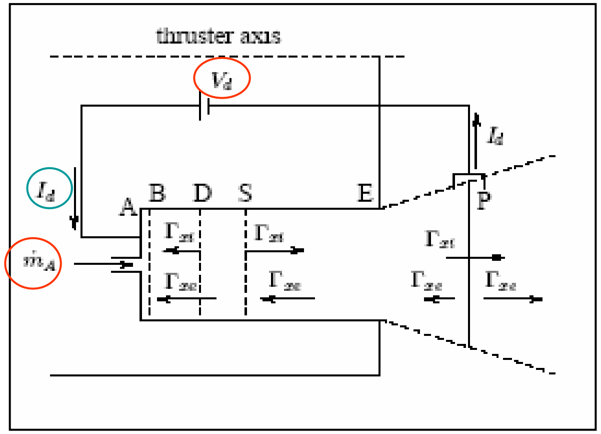
- For instance: $$\frac{du_{zi}}{dz} = \nu_i - u_{zi}\frac{d\ln A}{dz} - u_{zi}\frac{G}{T_e - m_i u_{zi}^2}$$

$$G = -\frac{\omega_e^2}{\nu_e} m_e u_{ze}\left(1 - \frac{2q_{ze}}{5n_e T_e u_{ze}}\right) - \nu_i m_i (2u_{zi} - u_{zn}) + m_i u_{zi}^2 \frac{d\ln A}{dz}$$

- A singular transition exists at anode sheath edge (point B): $M_B = -1$ & $G_B = \infty$
- A regular subsonic/supersonic transition exists inside the channel (point S) with

$$M_S = 1 \ \ \& \ \ \vec{F}(\vec{Y}_S) = \vec{0}, \quad \text{i.e.} \ \ G_S = 0$$

- Point S is equivalent to the critical section of a Laval nozzle, where $0 = G_S \equiv m_i u_{zi}^2 d\ln A/dz$ corresponds to $dA/dz = 0$. In a Hall thruster, $G_S = 0$ corresponds to a certain balance between ionization and electron diffusion.
- 8 boundary conditions are set at points B, S and P:
  - Discharge voltage (1)
  - Density and velocity of injected gas at anode (2)
  - Electron temperature at cathode (1)
  - Regularity condition at point S (1)
  - Anode sheath conditions (3)



Anode sheath in a Hall thruster

- For Maxwellian-type VDF electron flux to anode $\sim\ n_{eA}\sqrt{T_e/2\pi m_e}$,
- In normal conditions this is much larger than

the quasineutral flux in the channel, $g_{ze} \equiv n_e u_{ze} (>0)$

- A negative anode sheath AB is formed in order to

satisfy $g_{zeB} = g_{zeA}$: $\boxed{n_{eB} u_{zeB} \approx -n_{eB} \exp\left(-\frac{e\phi_{AB}}{T_{eB}}\right)\sqrt{\frac{T_{eB}}{2\pi m_e}}}$

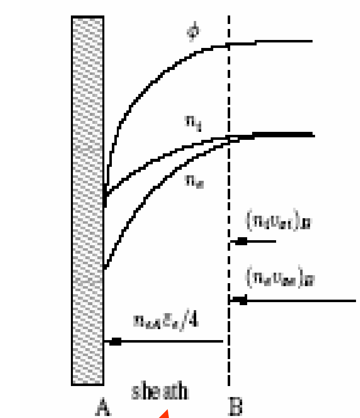
This equation determines the sheath potential fall, ϕ_{AB}.

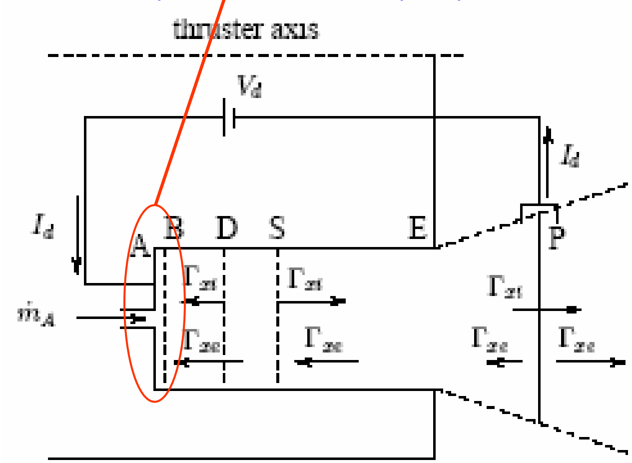
- Bohm condition at sheath edge: $\boxed{u_{ziB} \approx -\sqrt{T_{eB}/m_i}}$
- Electron energy flux deposited: a) at the anode A,

$$q_{zeA}^{tot} = -\int \frac{1}{2} m_e w^2 w_z f_{eB}(w) d^3 w \approx 2T_{eB} g_{zeB}$$

b) at the sheath edge B: $\boxed{q_{zeB}^{tot} = g_{zeB}(2T_{eB} + e\phi_{AB})}$

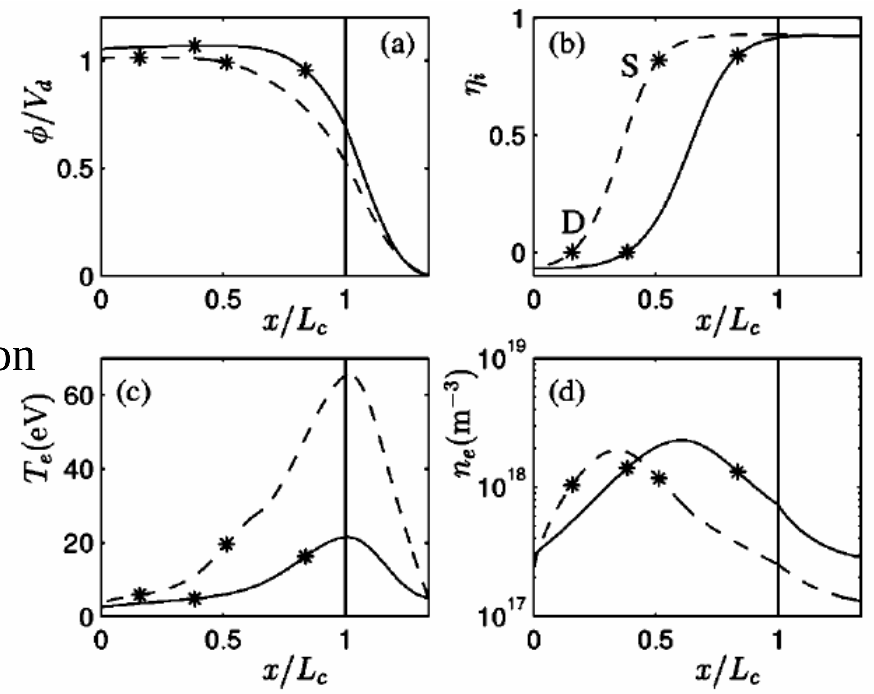
- These are 3 boundary conditions for the axial quasineutral model





Axial structure

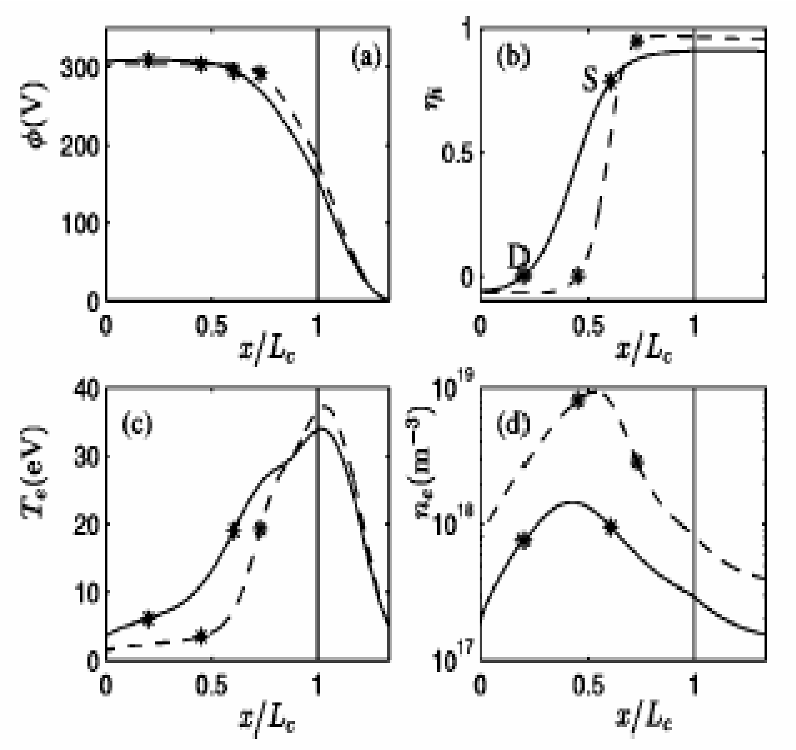
- **Acceleration region**:
  - Presents most of the potential drop & ion acceleration
  - For electrons: Joule heating competes with wall cooling
  - Heat conduction smooths T_e profile
  - Plasma production = plasma recombination
  - Plasma density decreases (due to ion acceleration)
- **Ionization region**:
  - Electron cooling due to ionization
  - Maximum of plasma density inside
- **Ion backstreaming region**:
  - Electric force very small
  - Ion reverse flow is small
  - Pressure drop (towards anode) dominates electron diffusive flow
  - Length depends on ionization rate (i.e. T_e)



- Magnetic field is adjusted for each case:
  - solid lines: 110V, 110G
  - dashed lines: 600V, 330G

Axial structure

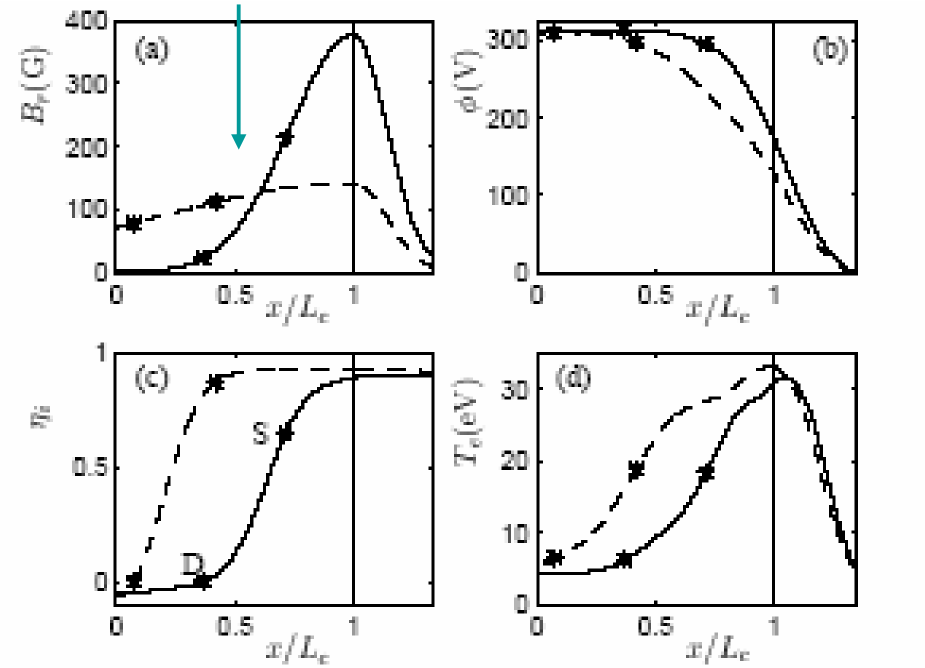
Influence of anode mass flow



- Solid lines: 4 mg/s
- Dashed lines: 10 mg/s

Influence of magnetic field shape

External B-field profiles



– solid lines: $L_{b1} = 9.5$mm, $B_{max} \sim 380$ G

– dashed lines: $L_{b1} = 30$mm, $B_{max} \sim 140$G

Thrust

- Plasma momentum flow: $F_p(z) = \sum_{\alpha=i,e,n} F_\alpha(z), \quad F_\alpha(z) = (m_\alpha u_{z\alpha}^2 + T_\alpha)\, n_\alpha A$
- Axial momentum equation for the plasma:

$$\frac{dF_i}{dz} = -Aen_i \frac{d\phi}{dz} + Am_i n_e (\nu_i u_{zn} - \nu_w u_{zi})$$

Electric force

Ionization + wall recombination

$$\frac{dF_n}{dz} = Am_i n_e (\nu_w u_{znw} - \nu_i u_{zn})$$

Magnetic force

$$\frac{dF_e}{dz} = Aen_e \frac{d\phi}{dz} + Aj_{\theta e} B_r + n_e T_e \frac{dA}{dz}$$

Jet divergence

$$\Rightarrow \boxed{\frac{dF_p}{dz} = \frac{\varepsilon_0}{2} \frac{d}{dz}\left(\frac{d\phi}{dz}\right)^2 + Aj_{\theta e} B_r - A\nu_w m_i (u_{zi} - u_{znw}) n_e + n_e T_e \frac{dA}{dz}}$$

Non-neutral E-field force

Thrust

- Integrating the preceding equation between anode and far downstream:
- Thrust: $\boxed{F = F_{p\infty} - D_{plume} = F_{mag} + F_{pA} - F_{elec} - D_{wall}}$
  - Magnetic force: $F_{mag} = \int_A^\infty j_{\theta e} B_r A dz$
  - Electric force: $F_{elec} = \left[A \frac{\varepsilon_0}{2} \left(\frac{d\phi}{dz} \right)^2 \right]_A$
  - Ion wall impact drag: $D_{wall} = \int_A^E \nu_w m_i (u_{zi} - u_{znw}) n_e A dz$
  - Jet divergence drag: $D_{plume} = \int_A^\infty n_e T_e \frac{dA}{dz} dz$
- Therefore, thrust is transmitted to the thruster through the magnetic reaction force of electrons on the thruster magnetic circuit.
  - Notice the contribution of the external magnetic field to thrust.
  - Pressure forces at the anode make a marginal contribution
  - Ion energy accommodation at walls acts as a drag force.

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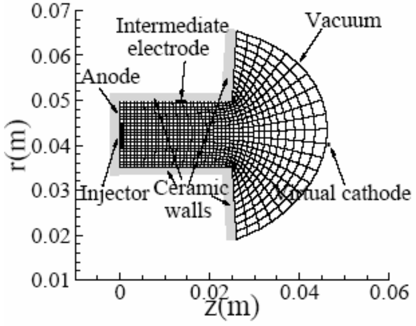
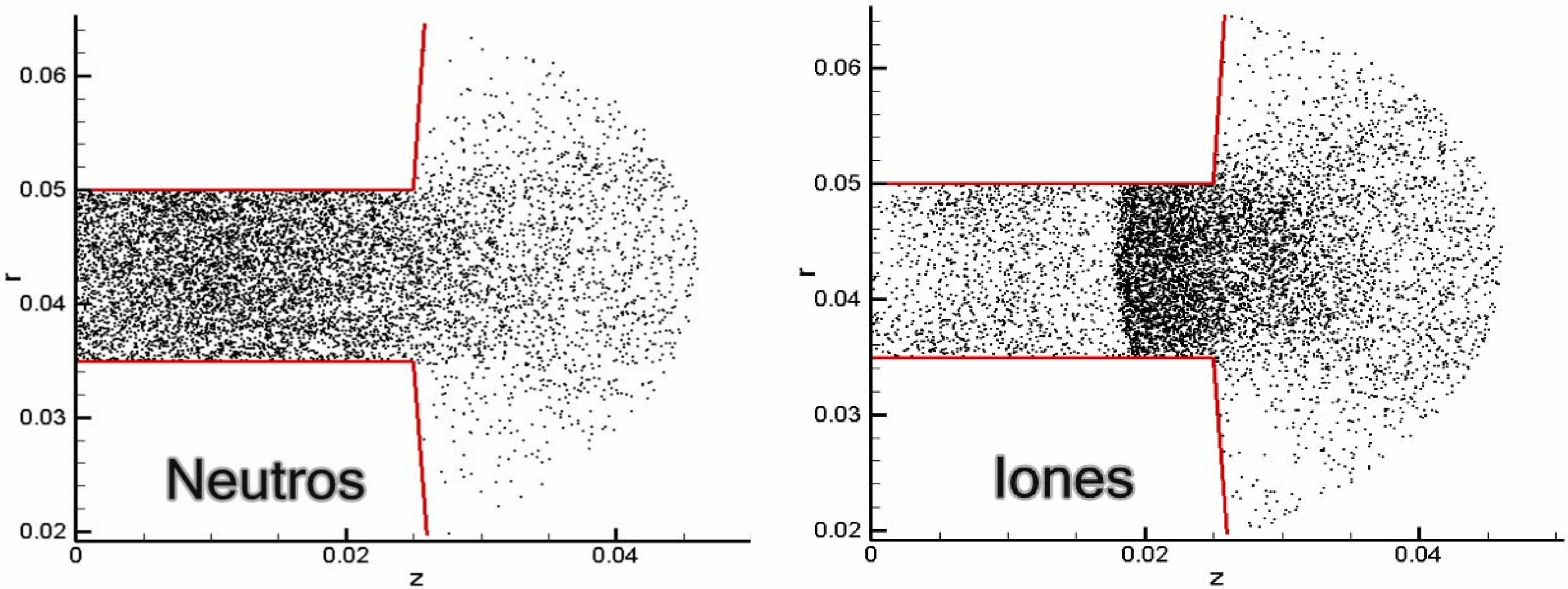
HALL THRUSTERS: HYBRID 2D CODE

Particle-in-cell (PIC) methods

- Individual motion of macroparticles, subject to electromagnetic forces, is followed in a computational grid. (see Birdsall-Langdon)
- Cell size, l_{cell}, smaller than plasma gradients
- Timestep, $\Delta\tau$, such that particles advance less than cell size.
- Number of macroparticles per cell, N_{cell}, depends on good statistics and small numerical oscillations.
- Mass of macroparticle depends on species density
- Different species → different sets of macroparticles
- Example for Hall thruster (only ions and neutrals)
  - axisymmetric, 30mm × 15 mm → 900 (toroidal) cells → l_{cell}~0.5mm
  - N_{cell} (per species) ~ 30 → 27.000 macroparticles/species → 10^9 -10^{11} atoms/macroparticle
  - macroparticle mass ~ atom mass × (atom/macroparticle)
  - $\Delta\tau \sim 10^{-8}$ s (ions), 10^{-7} s (neutrals), 10^{-10} s (electrons)



Video of particle motion



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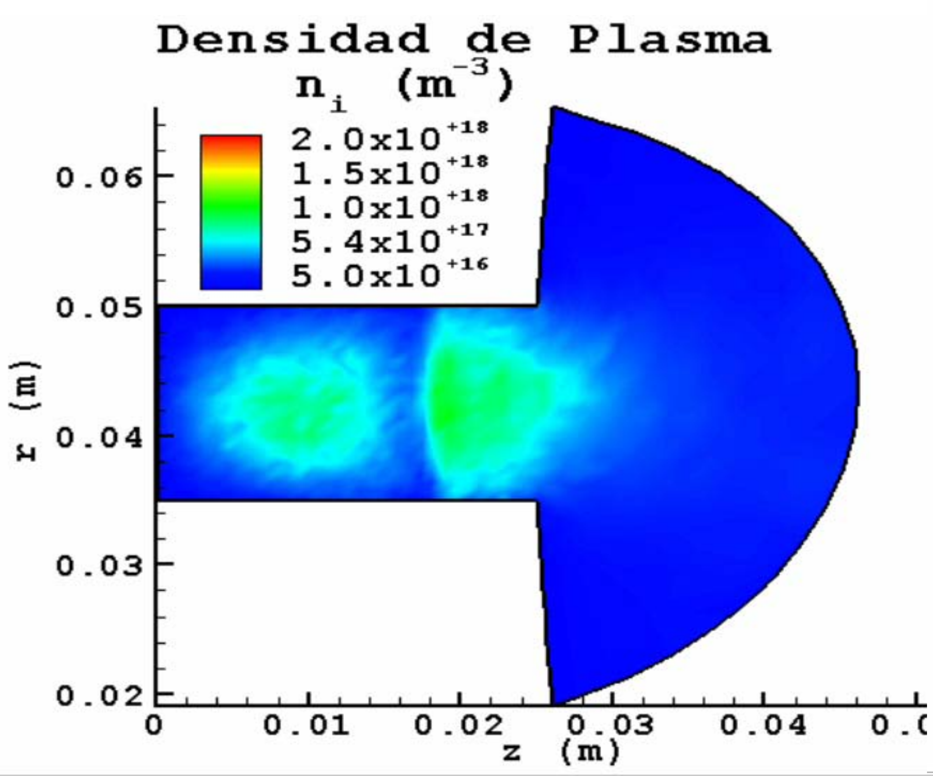
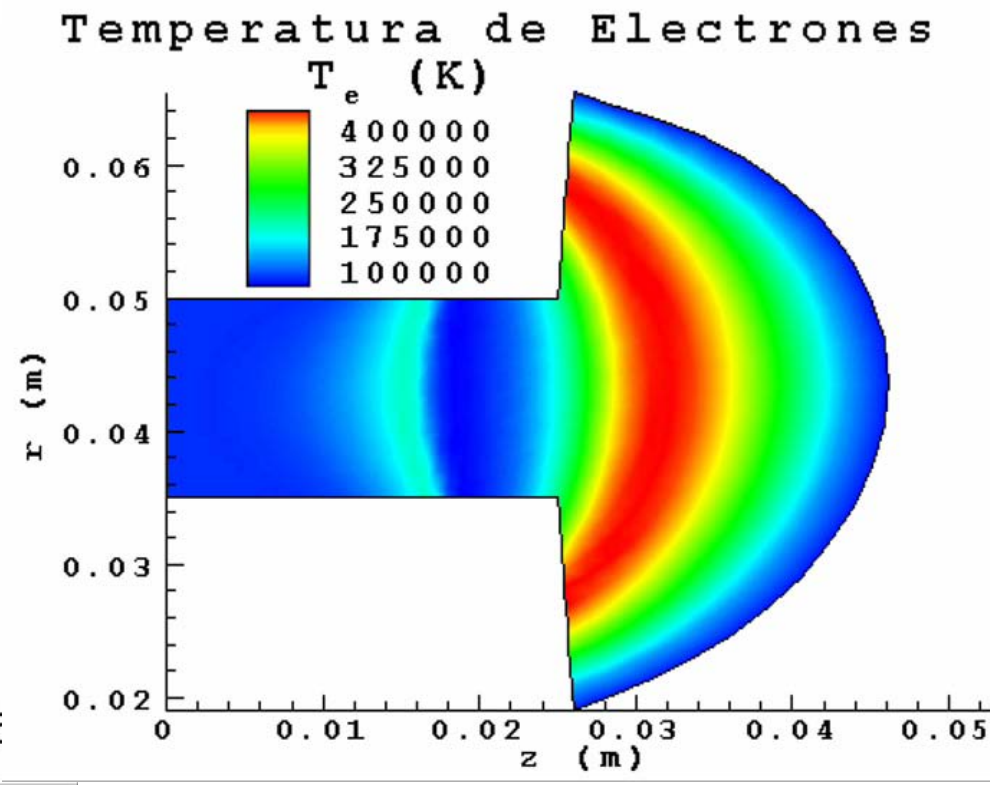
Video of plasma dynamics





Time-averaged 2D behavior

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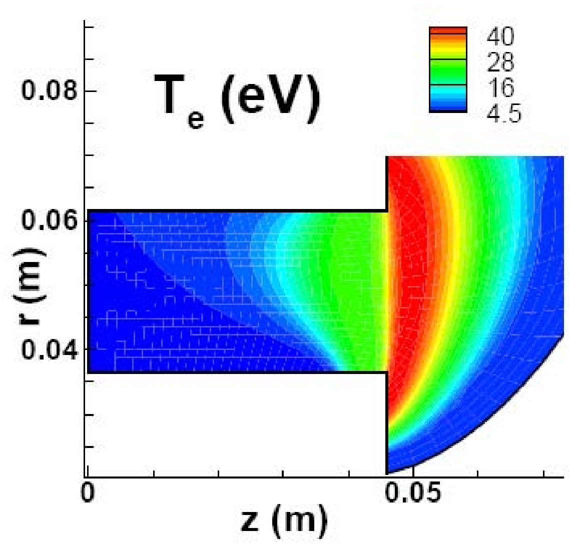
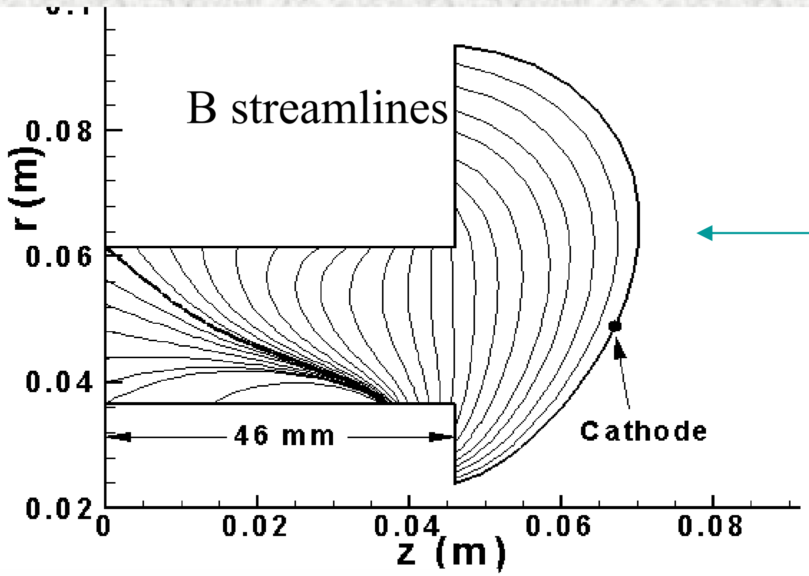
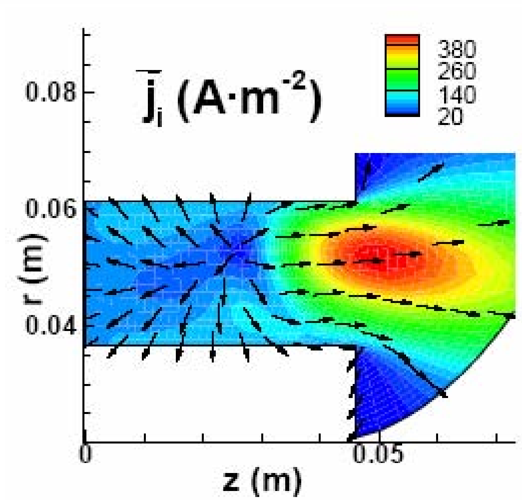
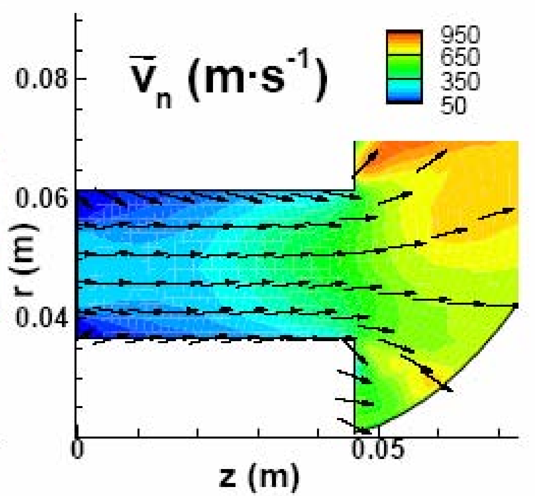
Time-averaged 2D behavior





T_e=const along magnetic lines





Ion recombination at walls & neutral re-emission

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