Lecture 15: Spacecraft Attitude Dynamics

Goal: Gain understanding of the physical dynamics of the attitude motions of spacecraft
Introduce S/C attitude control

Spacecraft Motion:

Position + Linear Velocity $\Rightarrow$ Orbital Motion

Equations of motion governed by the gravitational field

Planetary scale (earth, planets, moon, etc.) is large compared to the S/C size

S/C orbital motion is independent of S/C attitude

Orbital motion can be determined independently of S/C attitude motions.

Attitude Dynamics Analysis

Determine (predict) the attitude motion of S/C

Attitude Control

Process of orienting S/C

Stabilization - maintain desired orientation

Maneuver control - attitude change

Important for -
- shading
- heat dissipation
- directing sensors
- orienting thrusters

Typical Requirements

± 1 deg. of attitude control for engineering functions

± fractions of seconds of arc attitude control for science

Given: Spacecraft on some trajectory with zero Thrust (only force is gravity)

Find: Equations of motion governing attitude motion.

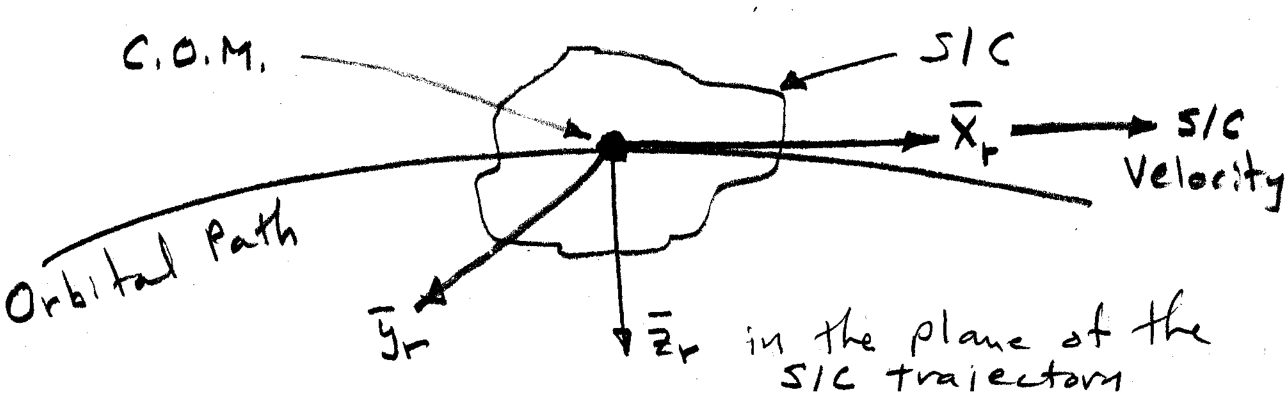
Assumption: S/C is a rigid body

Six degrees of freedom (3 linear, 3 angular)

Need six equations of motion

$\bar{F} = m\dot{\bar{V}} \qquad \bar{M} = \dot{\bar{H}}$

Key Point: Translational and rotational EOMs decouple if all EOMs are written with respect to the center of mass.



<u>R Frame</u> - Fixed at S/C center of mass

$\bar{x}_r \equiv$ aligned with S/C velocity w.r.t. inertial space

$\bar{z}_r \equiv$ in the plane of the trajectory and orthogonal to $\bar{x}_r$

$\bar{y}_r \equiv$ orthogonal to $\bar{x}_r$ and $\bar{z}_r$

Right hand rule

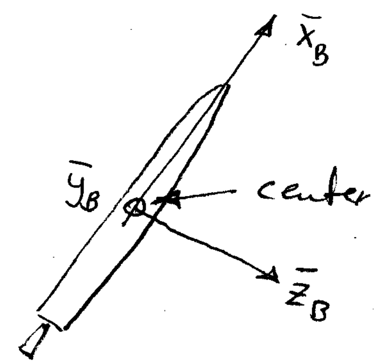
$$\bar{x}_r \times \bar{y}_r = \bar{z}_r$$

<u>Assumption</u>: Earth is fixed w.r.t. inertial space

Note that the R frame moves along the trajectory with the S/C. Its origin is always at the center of mass of the S/C, $\bar{x}_r$ always tangent to the trajectory path and z_r in the plane of the trajectory

${}^{i}\bar{\omega}^{R}$ = Angular velocity of R frame with respect to inertial space (earth fixed) frame.

B Frame - S/C body frame, with its origin at the center of mass of the S/C. Also, choose the axes of this frame to be principal axes of the S/C



${}^{R}\vec{\omega}^{B}$ = Angular velocity of the S/C (B frame) with respect to the R frame

Then the angular velocity of the S/C (B frame) with respect to inertial space is

$$ {}^{i}\bar{\omega}^{B} = {}^{i}\bar{\omega}^{R} + {}^{R}\bar{\omega}^{B} $$

S/C w.r.t. R frame

R frame w.r.t. inertial space

Now choose a frame to work in

Let's use the S/C (B frame)

We will coordinatize vectors in the B frame

We are interested in the rotational dynamics of the spacecraft. These are governed by the equation for the rate of change of its angular momentum

$$\bar{M} = \frac{d\bar{H}}{dt}^{i}$$

$\bar{M}$ = All external moments acting on the S/C. Typically control inputs from attitude control thrusters, momentum wheels, etc.

Now

$$\frac{d\bar{H}}{dt}^{i}_{B} = \frac{d\bar{H}}{dt}^{B}_{B} + {}^{i}\bar{\omega}^{B}_{B} \times \bar{H}_{B}$$

where

$$\frac{d\bar{H}}{dt_B}^{i} = \text{rate of change of } \bar{H} \text{ with respect to inertial space, coordinatized in the B frame}$$

$$\frac{d\bar{H}}{dt_B}^{B} = \text{rate of change of } \bar{H} \text{ with respect to the body, coordinatized in the B frame}$$

$$\omega_B^{i-B} = \text{angular velocity of the body with respect to inertial space, coordinatized in the B frame}$$

$$\bar{H}_B = \text{angular momentum vector, coordinatized in the body frame}$$

Define ω_x, ω_y and ω_z as the angular velocities of the body w.r.t. inertial space, coordinatized in the body frame.

$$\bar{\omega}_B^{i-B} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

so

$$\bar{H}_B = [I]\,{}^{i}\bar{\omega}^{B}_{B}$$

But since we have chosen the body axes to lie along the principal axes of the body, the inertia matrix is diagonal and

$$\bar{H}_B = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} I_{xx}\,\omega_x \\ I_{yy}\,\omega_y \\ I_{zz}\,\omega_z \end{bmatrix}$$

And in the body frame

$$\frac{d}{dt}\bar{H}^{B}_{B} = \begin{bmatrix} I_{xx}\,\dot{\omega}_x \\ I_{yy}\,\dot{\omega}_y \\ I_{zz}\,\dot{\omega}_z \end{bmatrix}$$

So

$$\bar{M}_B = \frac{d}{dt}\bar{H}^{i}_{B} = \begin{bmatrix} I_{xx}\,\dot{\omega}_x \\ I_{yy}\,\dot{\omega}_y \\ I_{zz}\,\dot{\omega}_z \end{bmatrix} + \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \begin{bmatrix} I_{xx}\,\omega_x \\ I_{yy}\,\omega_y \\ I_{zz}\,\omega_z \end{bmatrix}$$

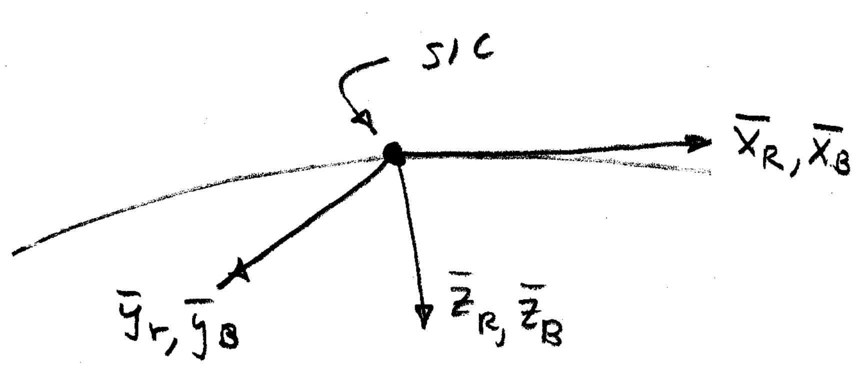
So, finally, we have the EOMs in body coordinates

$$\boxed{\begin{aligned} M_x &= I_{xx}\dot{\omega}_x + (I_{zz} - I_{yy})\,\omega_y\omega_z \\ M_y &= I_{yy}\dot{\omega}_y + (I_{xx} - I_{zz})\,\omega_z\omega_x \\ M_z &= I_{zz}\dot{\omega}_z + (I_{yy} - I_{xx})\,\omega_x\omega_y \end{aligned}}$$

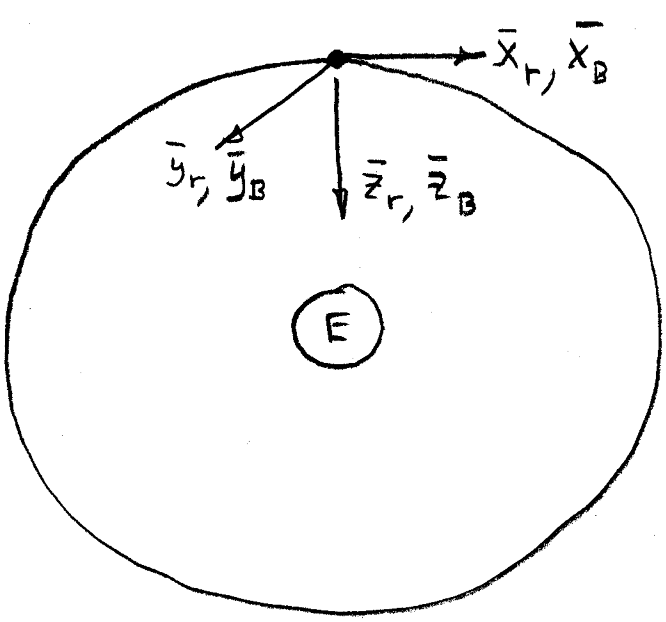
These are Euler's Equations

Nonlinear, coupled differential equations.

Let's consider a special case where we want to keep the S/C (B frame) aligned with the trajectory or orbit frame (R frame)



Suppose the S/C trajectory is a circular earth orbit of period T



$T \cong 90$ minutes.

Then the nominal angular velocity of the R frame is

$$ {}^{i}\bar{\omega}_B^{\,r} = \begin{bmatrix} 0 \\ -\omega_0 \\ 0 \end{bmatrix} $$

where

$$ \omega_0 = \frac{2\pi}{T} \cong \frac{2 \times 3.14}{90 \times 3600} \cong 2 \times 10^{-5} \text{ rad/sec.} $$

Lets further assume that the s/c attitude control system can maintain the s/c close to the desired attitude (i.e. the B frame close to the R frame)

Then $${}^{r}\overline{\omega}_{B}^{B} \cong \begin{bmatrix} \delta\omega_x \\ \delta\omega_y \\ \delta\omega_z \end{bmatrix}$$ ← small angular velocities about the three body axes.

Hence
$${}^{i}\overline{\omega}_{B}^{B} = {}^{i}\overline{\omega}_{B}^{R} + {}^{R}\overline{\omega}_{B}^{B} = \begin{bmatrix} \delta\omega_x \\ \delta\omega_y - \omega_o \\ \delta\omega_z \end{bmatrix} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

We can then substitute back into Euler's equation

$$M_x = I_{xx}\,\delta\dot{\omega}_x + (I_{zz} - I_{yy})(\delta\omega_y - \omega_o)\,\delta\omega_z$$

$$M_y = I_{yy}\,\delta\dot{y} + (I_{xx} - I_{zz})\,\delta\omega_z\,\delta\omega_x$$

$$M_z = I_{zz}\,\delta\dot{z} + (I_{yy} - I_{xx})\,\delta\omega_x\,(\delta\omega_y - \omega_o)$$

We linearize these equations by dropping products of small terms. Rearranging yields

$$\delta\dot{\omega}_x = \frac{(I_{zz} - I_{yy})}{I_{xx}}\,\omega_0\,\delta\omega_z + \frac{M_x}{I_{xx}}$$

$$\delta\dot{\omega}_y = \frac{M_y}{I_{yy}}$$

$$\delta\dot{\omega}_z = \frac{(I_{yy} - I_{xx})}{I_{zz}}\,\omega_0\,\delta\omega_x + \frac{M_z}{I_{zz}}$$

Let's write these equations in terms of state vectors where

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}$$

$$\bar{x} = \begin{bmatrix} \delta\omega_x \\ \delta\omega_y \\ \delta\omega_z \end{bmatrix} \qquad \bar{u} = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$

so

$$A = \begin{bmatrix} 0 & 0 & \left(\frac{I_{zz} - I_{yy}}{I_{xx}}\right)\omega_0 \\ 0 & 0 & 0 \\ \left(\frac{I_{yy} - I_{xx}}{I_{zz}}\right)\omega_0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{1}{I_{xx}} & 0 & 0 \\ 0 & \frac{1}{I_{yy}} & 0 \\ 0 & 0 & \frac{1}{I_{zz}} \end{bmatrix}$$

Consider the stability of this system

If we define

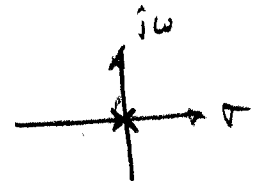
$$k_1 = \frac{I_{yy} - I_{zz}}{I_{xx}} \qquad k_3 = \frac{I_{yy} - I_{xx}}{I_{zz}}$$

then identity matrix

$$|sI - A| = \begin{vmatrix} s & 0 & k_1\omega_0 \\ 0 & s & 0 \\ -k_3\omega_0 & 0 & s \end{vmatrix} = s^3 + k_1 k_3 \omega_0^2 s$$

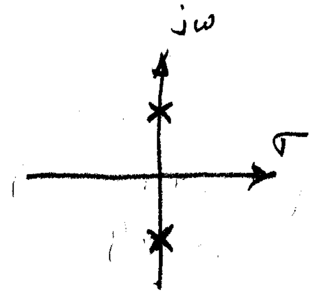
And the system characteristic equation is –

$$s(s^2 + k_1 k_3) = 0$$

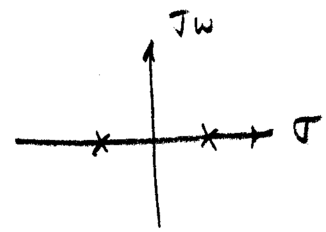


one pole at $s = 0$

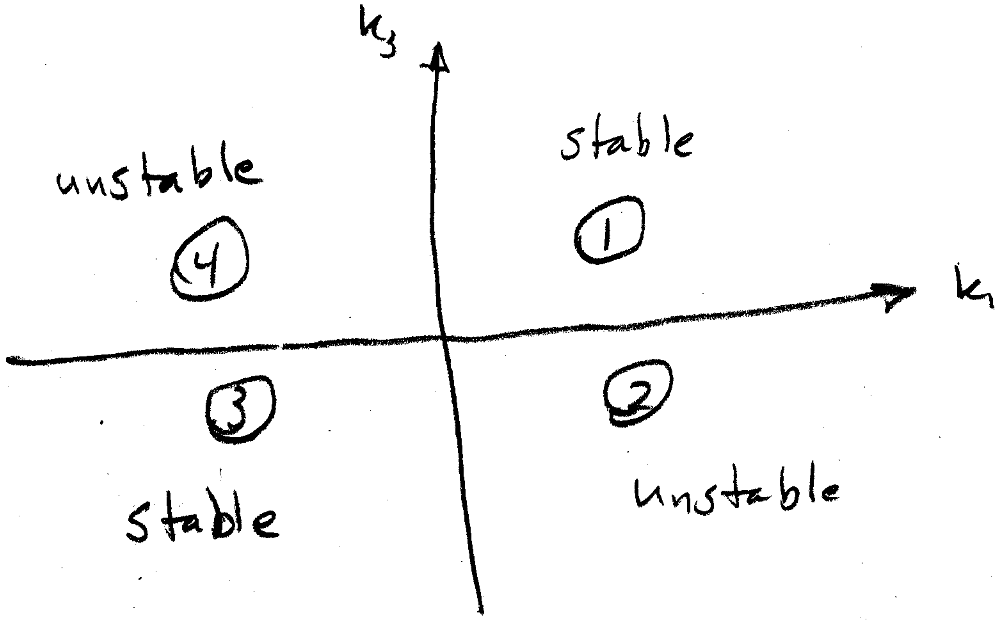
two poles at $s = \pm\sqrt{k_1 k_3}\, j$



Neutrally stable if $k_1 k_3 > 0$



Unstable if $k_1 k_3 < 0$



Consider each region in the k_1, k_3 space

Region 1

$k_1 > 0$, $k_2 > 0$

$I_{yy} > I_{zz}$ $\quad$ $I_{yy} > I_{xx}$

Rotation axis (y axis) is axis of maximum moment of inertia.

Stable (neutrally stable) motion !

Region 2

$k_1 > 0$, $k_3 < 0$

$I_{yy} > I_{zz}$ $\quad$ $I_{yy} < I_{xx}$

So

$I_{xx} > I_{yy} > I_{zz}$

Rotation axis (y axis) is axis of intermediate moment of inertia

Unstable motion !

Region 3

$k_1 < 0$, $k_3 < 0$

$I_{yy} < I_{zz}$ $\quad$ $I_{yy} < I_{xx}$

Rotation axis is axis of

minimum moment of inertia

stable motion!

Region 4

$k_1 < 0$, $k_3 > 0$

$I_{yy} < I_{zz}$ $\quad$ $I_{yy} > I_{xx}$

Similar to region 2

Unstable motion!

Stable motion if max or min moment of

inertia is the y_B axis

Spacecraft Control - Attitude Hold

The state equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -k_1\omega_0 \\ 0 & 0 & 0 \\ k_3\omega_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{I_{xx}} & 0 & 0 \\ 0 & \frac{1}{I_{yy}} & 0 \\ 0 & 0 & \frac{1}{I_{zz}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

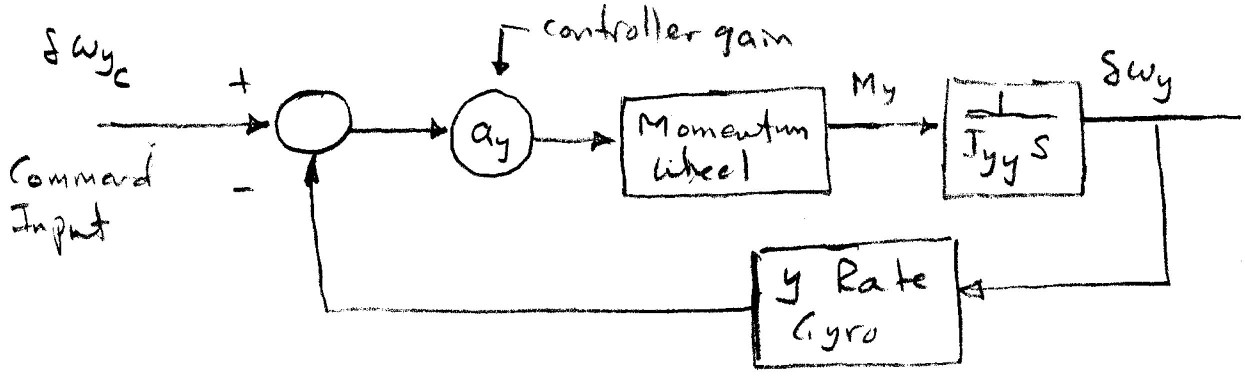
where $x_1 = \delta\omega_x \qquad x_2 = \delta\omega_y \qquad x_3 = \delta\omega_z$

$u_1 = M_x \qquad u_2 = M_y \qquad u_3 = M_z$

The x_2 equation is decoupled from the other two equations

$$\dot{x}_2 = \frac{u_2}{I_{yy}} \Rightarrow \delta\dot{\omega}_y = \frac{M_y}{I_{yy}} \Rightarrow \frac{\delta\omega_y(s)}{M_y(s)} = \frac{1}{I_{yy}s}$$

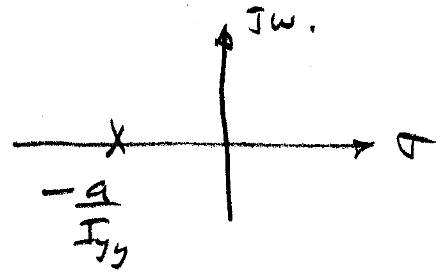
This is a simple first order system. A proportional controller would work



$$\frac{\delta \omega_y}{\delta \omega_{y_c}} = \frac{1}{\frac{I_{yy}}{a} s + 1}$$

First order system with time constant -

$$\tau = \frac{I_{yy}}{a}$$



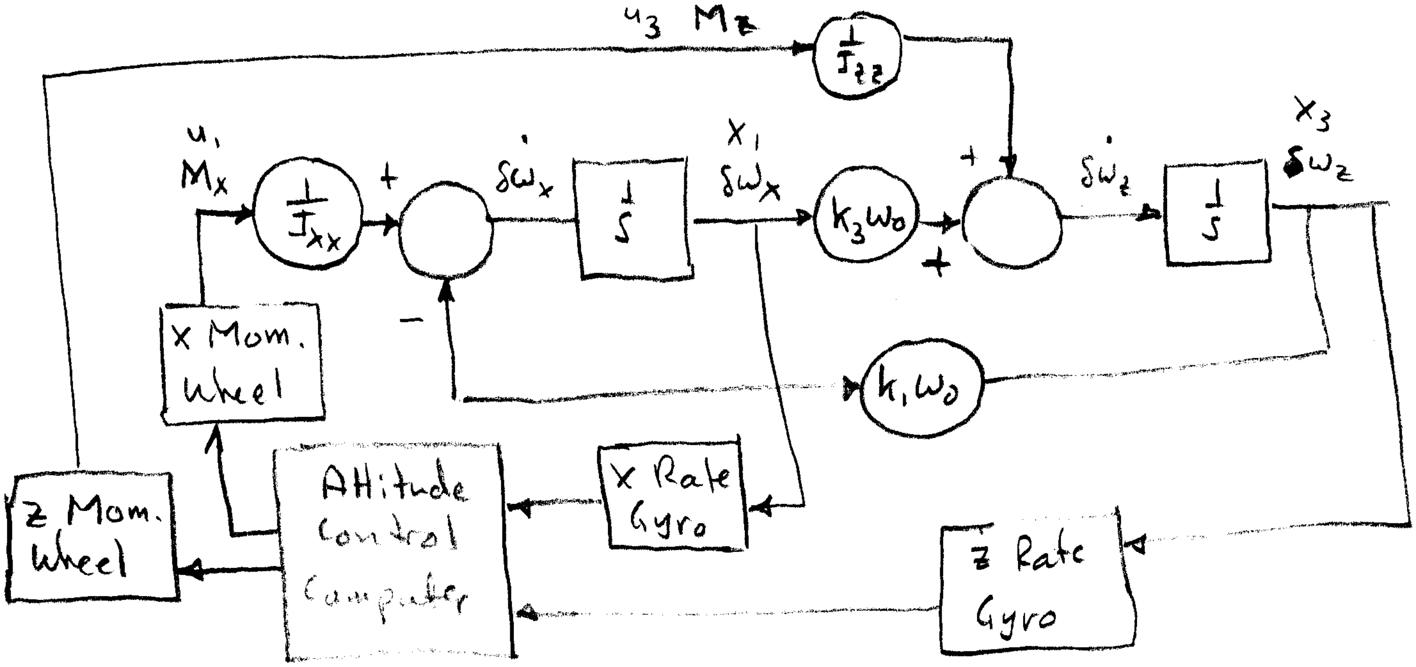
Choose a to achieve desired performance.

We could also use a proportional plus integral controller to eliminate steady state errors.

For the other two states we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & -k_1 \\ k_3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{I_{xx}} & 0 \\ 0 & \frac{1}{I_{yy}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_3 \end{bmatrix}$$

Typically there are rate gyros for all three axes so we would have full state feedback and we could do pole assignment to achieve a desired closed loop system. It might look like this

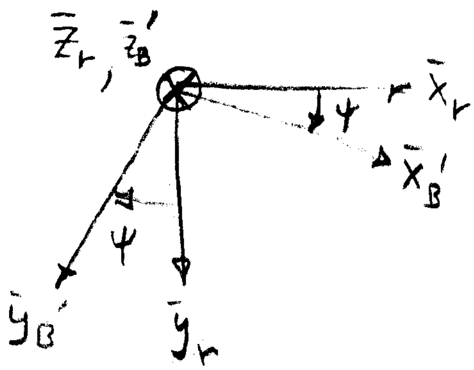


$$M_x = c_1\,\delta\omega_x + c_2\,\delta\omega_z$$

$$M_z = c_3\,\delta\omega_x + c_4\,\delta\omega_z$$

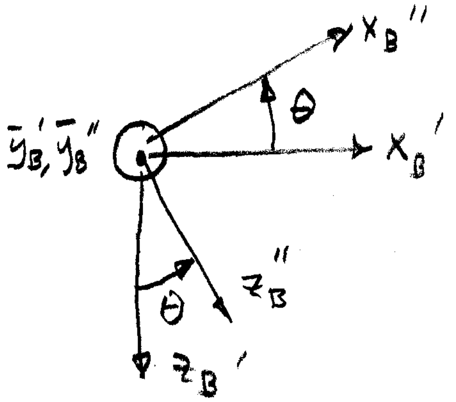
Spacecraft Control - Attitude Maneuvers

When we wish to change S/C attitude in some significant way the control strategy is different. We must now consider large angle effects. Define roll, pitch and yaw angles as three ordered rotations. First is yaw about the z axis



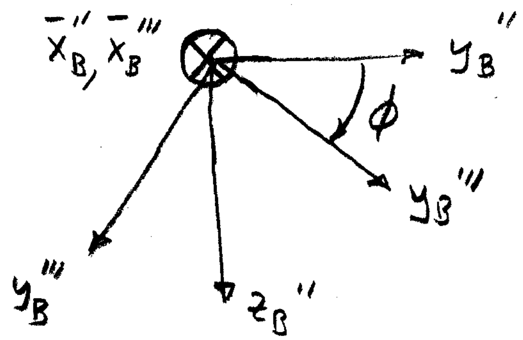
$$T_3(\psi) = \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Second is pitch about the y_B' axis



$$T_2(\theta) = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix}$$

And third is roll about the x_B'' axis



$$T_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{bmatrix}$$

Now suppose we have rates of change of these three angles

$\dot{\psi}$ about the $\bar{z}_r, \bar{z}_B'$ axis
$\dot{\theta}$ about the $\bar{y}_b', \bar{y}_b''$ axis
$\dot{\phi}$ about the x_b'', x_b''' axis

To obtain ${}^{r}\bar{\omega}_{B}^{B}$ we have

$$
{}^{r}\bar{\omega}_{B}^{B} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + T_1(\phi) \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + T_1(\phi) T_2(\theta) \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}
$$

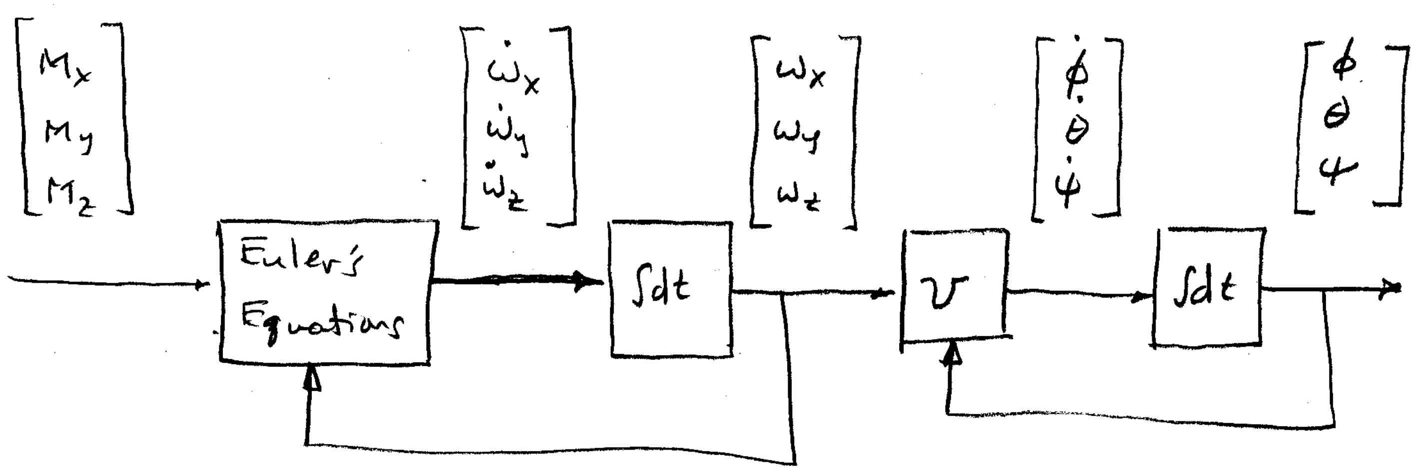
which yields

$$
\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & -s\theta \\ 0 & c\phi & c\theta\, s\phi \\ 0 & -s\phi & c\theta\, c\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}
$$

The inverse relationship is

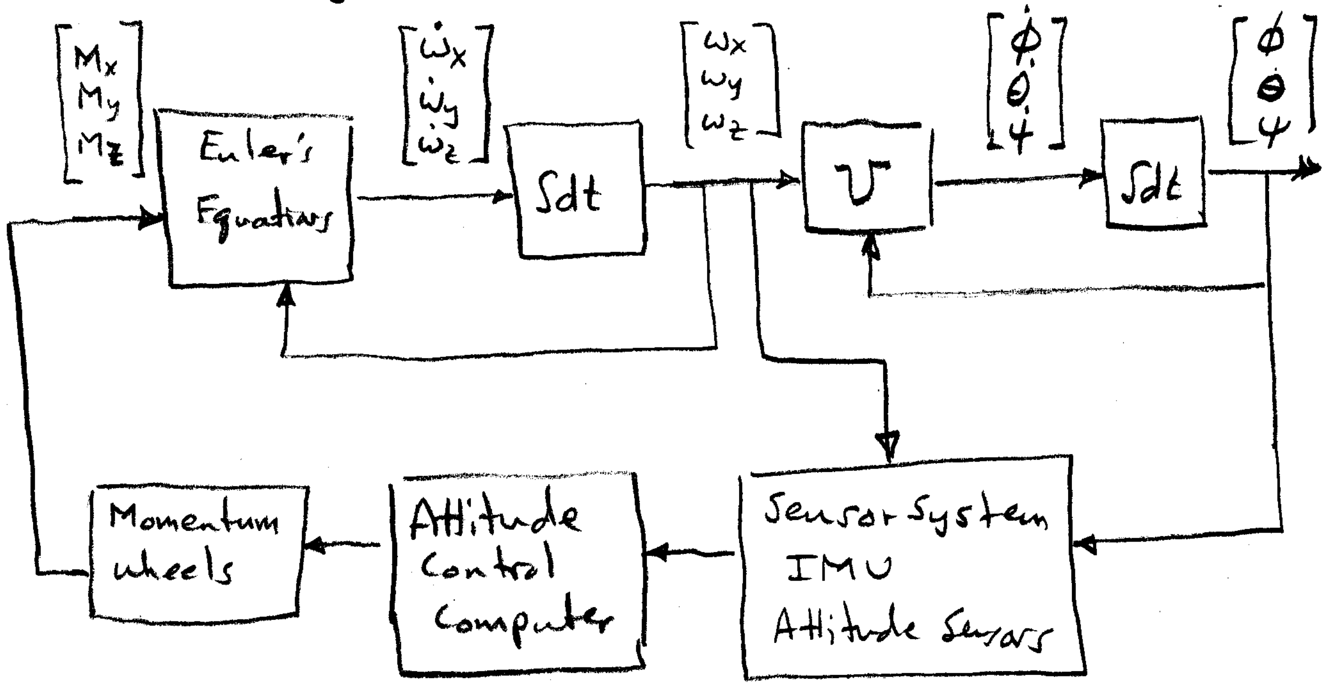
$$
\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \frac{s\phi\, s\theta}{c\theta} & \frac{c\phi\, s\theta}{c\theta} \\ 0 & c\phi & -s\phi \\ 0 & \frac{s\phi}{c\theta} & \frac{c\phi}{c\theta} \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}}_{{}^{r}\bar{\omega}^{b}}
$$

Thus we have the following system



Both Euler's equations and the $\mathcal{V}$ matrix are nonliner. Typically a S/C would have an inertial system and external sensors (e.g. star trackers) to determine the angular velocities and the attitude. The system has six state variables $\phi, \theta, \psi, \omega_x, \omega_y, \omega_z$, all of which are measured by the sensors and can be fed back to a controller

The system architecture would be thus



Attitude Control computer determines a control strategy to get from some initial point to some final point

$$\begin{bmatrix} \phi_1 \\ \theta_1 \\ \psi_1 \end{bmatrix} \longrightarrow \begin{bmatrix} \phi_2 \\ \theta_2 \\ \psi_2 \end{bmatrix}$$

$$\bar{\omega}_1 \longrightarrow \bar{\omega}_2$$

Highly nonlinear problem. Typically involves solution of a two point boundary value problem to define a trajectory in the six dimensional state space