Ground-Based Testbed for Replicating the Orbital Dynamics of a Satellite Cluster in a Gravity Well

David W. Miller
Raymond J. Sedwick
AFRL Distributed Satellite Systems Program
MIT Space Systems Laboratory

Hill's Equations

- Governing equations where 'n' is orbital frequency in rad/sec:

$$\begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{Bmatrix} + \begin{bmatrix} 0 & -2n & 0 \\ 2n & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix} + \begin{bmatrix} -3n^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n^2 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}$$

  - accelerations account for non-central forces (drag, thrust, etc.).
  - x-axis in zenith, y-axis in frame's velocity, and z-axis in transverse directions.

- Free orbit solution where 'A' and 'B' are lengths and 'α' and 'β' are phase angles.

$$\begin{aligned} x &= A\cos(nt+\alpha) + x_o \\ y &= -2A\sin(nt+\alpha) - (3/2)nx_o t + y_o \\ z &= B\cos(nt+\beta) \end{aligned}$$

Closed Cluster Solution

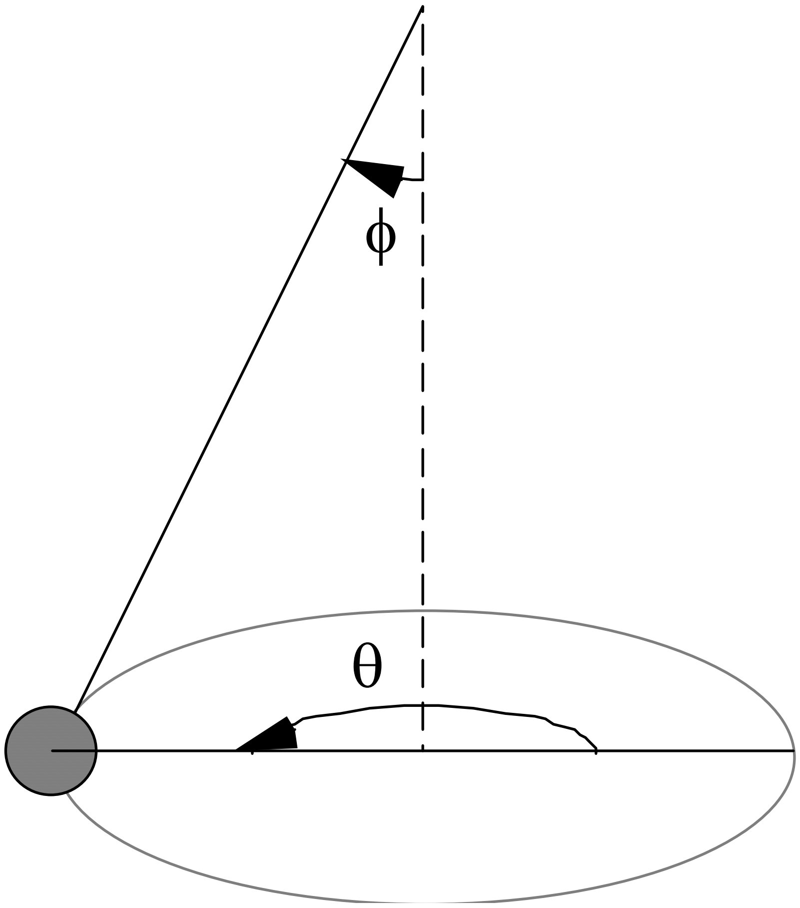
- There exist free orbits that cause a S/C to follow a closed and periodic motion with respect to the Hill's frame as well as other S/C of the same period.

$$
\begin{aligned}
x &= A\cos(nt+\alpha) \\
y &= -2A\sin(nt+\alpha)+y_o \\
z &= B\cos(nt+\beta)
\end{aligned}
$$

- the S/C must follow a two-by-one ellipse in the Hill's frame's zenith-velocity plane.
  - transverse displacement is independent and oscillatory.
- The parameters A, B, α, β, and y_o can be selected for each spacecraft in the cluster.
  - based upon the projection of some ground track motion.
  - to allow natural orbital dynamics to most uniquely sweep out aperture baselines.
  - to make the array appear "rigid" from some perspective.

Consider a Pendulum in 1-G

- Parameterize pendulum motion in terms of azimuth (θ) and elevation (ϕ) angles:



Dynamics of a Pendulum

- Define the Lagrangian as the difference between the kinetic and potential energies:

$$L = T - V = \frac{1}{2} m\left[(r\dot{\phi})^2 + (r\dot{\theta}\sin\phi)^2\right] - mgr\left[1 - \cos\phi\right]$$

- Nonlinear dynamic equations found using Lagrange's Equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \quad \text{where} \quad q = \text{generalized DOF}$$

- Results in the following equations

$$\left[\phi\right]: \qquad mr^2\ddot{\phi} - m(r\dot{\theta})^2 \sin\phi\cos\phi + mgr\sin\phi = 0$$

$$\left[\theta\right]: \qquad m(r\sin\phi)^2\ddot{\theta} + 2mr^2\dot{\theta}\dot{\phi}\sin\phi\cos\phi = 0$$

Perturbed Pendulum Motion

- Perturb motion about a nominal elevation angle and azimuthal angular rate:

$$\phi = \phi_o + \delta\phi \quad , \quad \dot{\theta} = \dot{\theta}_o + \delta\dot{\theta} \quad \text{where} \;\; \phi_o, \dot{\theta}_o = \text{const}$$

- Substitute into nonlinear equations and zero higher order terms:

$$[\phi]: \quad \delta\ddot{\phi} - \left[\dot{\theta}_o^2(\cos^2\phi_o - \sin^2\phi_o) - \frac{g}{r}\cos\phi_o\right]\delta\phi - 2\dot{\theta}_o \sin\phi_o \cos\phi_o \delta\dot{\theta}$$

$$= \left(\dot{\theta}_o^2 \cos\phi_o - \frac{g}{r}\right)\sin\phi_o$$

$$[\theta]: \quad \delta\ddot{\theta} + 2\dot{\theta}_o \frac{\cos\phi_o}{\sin\phi_o}\delta\dot{\phi} = 0$$

- Notice that forcing term zeroes about equilibrium motion:

$$\dot{\theta}_o^2 = \frac{1}{\cos\phi_o}\frac{g}{r}$$

Comparison with Hill's Equations

- Two DOF Linearized Pendulum Equations:

$$\begin{Bmatrix} \delta\ddot{\phi} \\ \delta\ddot{\theta} \end{Bmatrix} = \begin{bmatrix} 0 & 2\sqrt{\frac{g}{r}}\sin\phi_o\sqrt{\cos\phi_o} \\ -2\sqrt{\frac{g}{r}}\frac{\sqrt{\cos\phi_o}}{\sin\phi_o} & 0 \end{bmatrix} \begin{Bmatrix} \delta\dot{\phi} \\ \delta\dot{\theta} \end{Bmatrix} + \begin{bmatrix} -\frac{g}{r}\frac{\sin^2\phi_o}{\cos\phi_o} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \delta\phi \\ \delta\theta \end{Bmatrix}$$

- Evaluated at $\phi_o = 64^o$

$$\begin{Bmatrix} \delta\ddot{\phi} \\ \delta\ddot{\theta} \end{Bmatrix} = \begin{bmatrix} 0 & 1.8n \\ -2.2n & 0 \end{bmatrix} \begin{Bmatrix} \delta\dot{\phi} \\ \delta\dot{\theta} \end{Bmatrix} + \begin{bmatrix} -4.2n^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \delta\phi \\ \delta\theta \end{Bmatrix} \quad \text{where } n = \sqrt{\frac{g}{r}}\sqrt{\cos\phi_o}$$

- Two DOF Linearized Hill's Equations:

$$\begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} = \begin{bmatrix} 0 & 2n \\ -2n & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} + \begin{bmatrix} 3n^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

General Solutions: Secular & Periodic

- Pendulum Equations:

$$\delta\phi = A\cos(n\rho t + \alpha) + \delta\phi_o$$

$$\delta\theta = -\frac{2A}{\rho \sin\phi_o}\sin(n\rho t + \alpha) + \frac{n(\rho^2 - 4)}{2\sin\phi_o}\delta\phi_o t + \delta\theta_o$$

$$\text{where} \quad n = \sqrt{\frac{g}{r}}\sqrt{\cos\phi_o} \quad \text{and} \quad \rho = \sqrt{4 + \frac{\sin^2\phi_o}{\cos^2\phi_o}}$$

- Hill's Equations:

$$x = A\cos(nt + \alpha) + x_o$$

$$y = -2A\sin(nt + \alpha) - (3/2)nx_o t + y_o$$

Periodic Solutions

- Pendulum Equations:

$$\delta\phi = A\cos(n\rho t + \alpha)$$

$$\delta\theta = -\frac{2A}{\rho\sin\phi_o}\sin(n\rho t + \alpha) + \delta\theta_o$$

$$\text{where} \quad n = \sqrt{\frac{g}{r}\sqrt{\cos\phi_o}} \quad \text{and} \quad \rho = \sqrt{4 + \frac{\sin^2\phi_o}{\cos^2\phi_o}}$$

- Hill's Equations:

$$x = A\cos(nt + \alpha)$$

$$y = -2A\sin(nt + \alpha) + y_o$$

Eigenvalues

- Pendulum Equations:

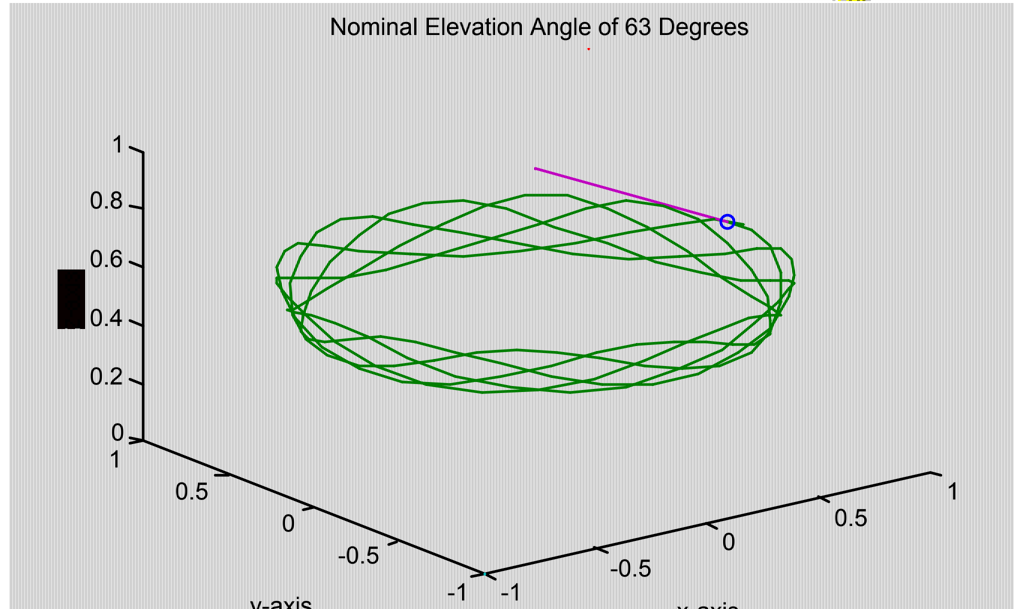
$$s = \pm in\sqrt{4 + \frac{\sin^2\phi_o}{\cos^2\phi_o}} \quad \text{where} \quad n = \sqrt{\frac{g}{r}}\sqrt{\cos\phi_o}$$

- Hill's Equations:

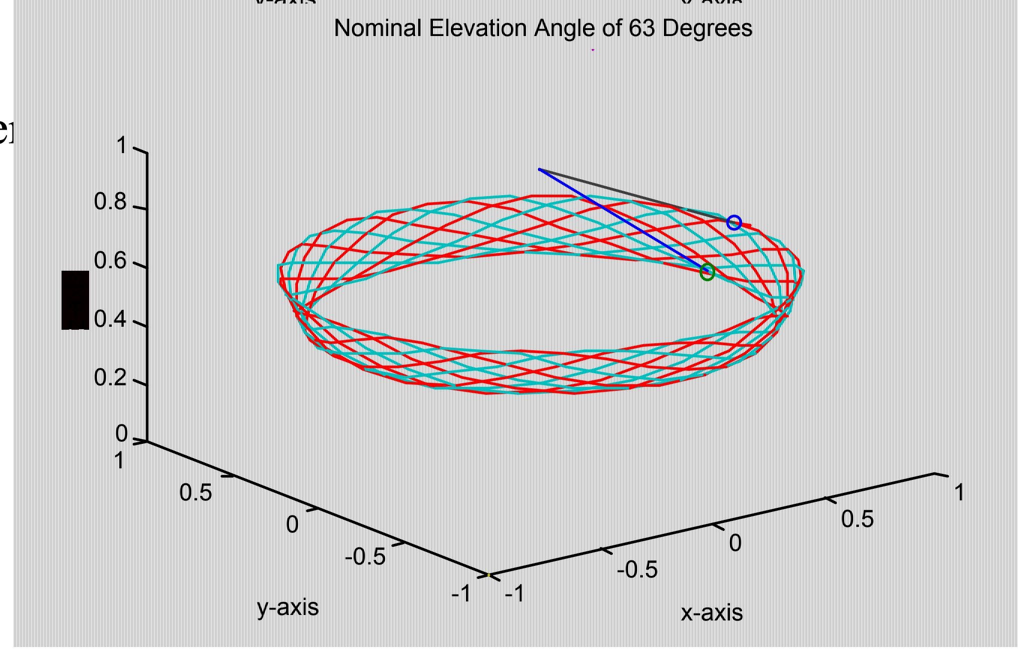
$$s = \pm in \qquad \text{where} \quad i = \sqrt{-1}$$

Perturbed Motion About 63 Degree Elevation

- Single pendulum system
  - at 63 degrees elevation, S/C oscillates slightly less than three cycles per revolution

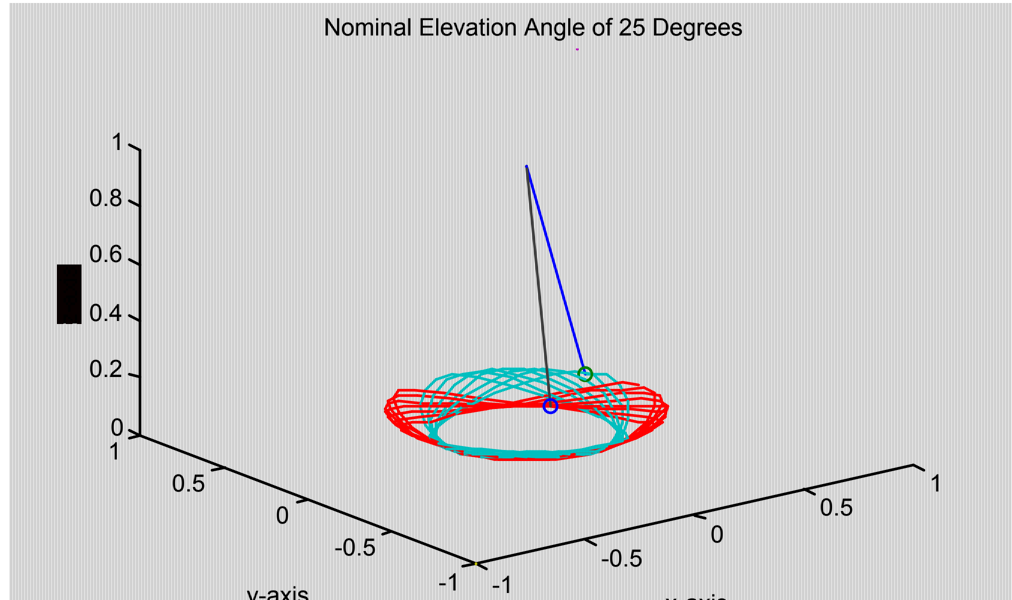


- Douple pendulum system
  - higher elevation S/C moves slowe and falls behind
  - lower elevation S/C moves faster and moves ahead
  - similar to Hill’s equations

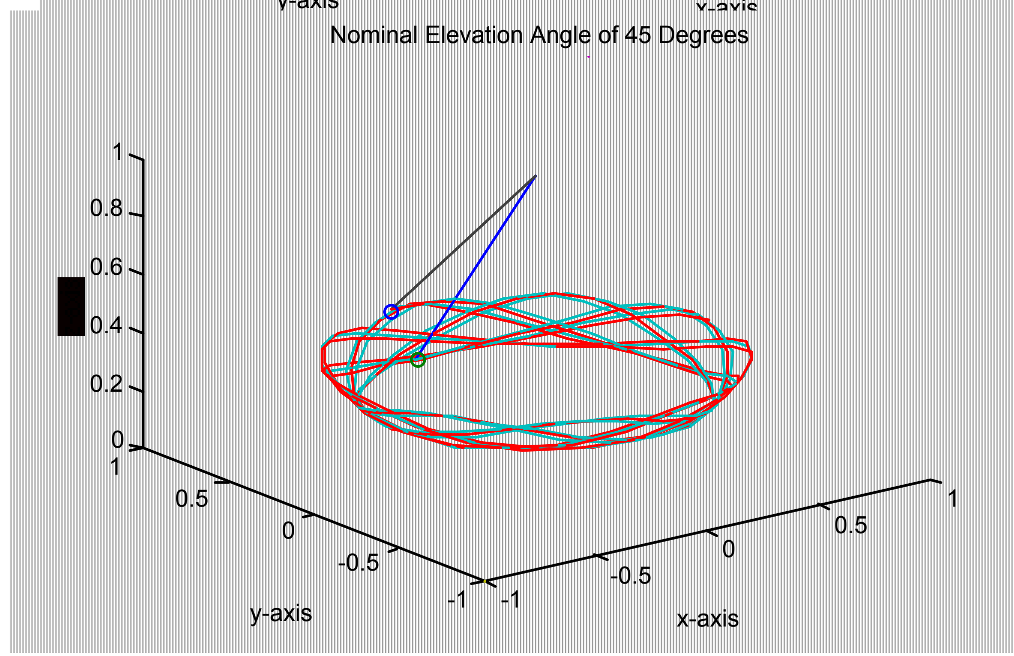


Perturbed Motion at Other Elevation Angles

- Elevation angle of 25 degrees
  - number of oscillations per revolution decreases with decreasing nominal elevation angle



- Elevation angle of 45 degrees
  - speed increases with increasing nominal elevation angle



Design Parameters

r (m)	ϕ_o (deg)	$n = \dot{\theta}_0$ (rad/s)	Circum (m)	Speed (m/s)	T (s)
10	25	1.040	26.55	4.40	6.03
	45	1.178	44.43	8.33	5.33
	63	1.470	55.98	13.10	4.27
	85	3.355	62.59	33.42	1.87
20	25	0.736	53.11	6.22	8.54
	45	0.833	88.86	11.78	7.54
	63	1.039	111.97	18.52	6.05
	85	2.372	125.19	47.26	2.65