

**Massachusetts Institute of Technology**  
**Biological Engineering Division**  
**Department of Mechanical Engineering**  
**Department of Electrical Engineering and Computer Science**

**BEH.410/2.798J/6.524J/10.539J**  
**Molecular, Cellular & Tissue Biomechanics, Spring 2003**

**Problem Set # 5 (Viscoelastic materials)**

**Issued: Mon., 3/17/03**

**Due: Mon., 4/7/03**

---

**READING ASSIGNMENT:**

Fung, *Biomechanics*

Ferry, *Viscoelastic Properties of Polymers*

Carew et al, "Quasi-Linear Viscoelastic Theory Applied to Aortic Leaflets," (ASME) J Biomech Eng, 1999.

---

**PROBLEMS:**

Do problems 1-3 attached.

Continued on next page....

## Problem 1: Maxwell Material

Experiments have been performed in which a magnetic bead is attached to a cell membrane via the fibronectin-integrin bond, and a force is applied via a time-varying magnetic field. The graph below shows the result of applying force to a bead attached to cell receptors, expressed in terms of the bead displacement as a function of time (“Experimental Data”). The force increases linearly with time according to:

$$F(t) = 250t \text{ pN}$$

where  $t$  is expressed in seconds.

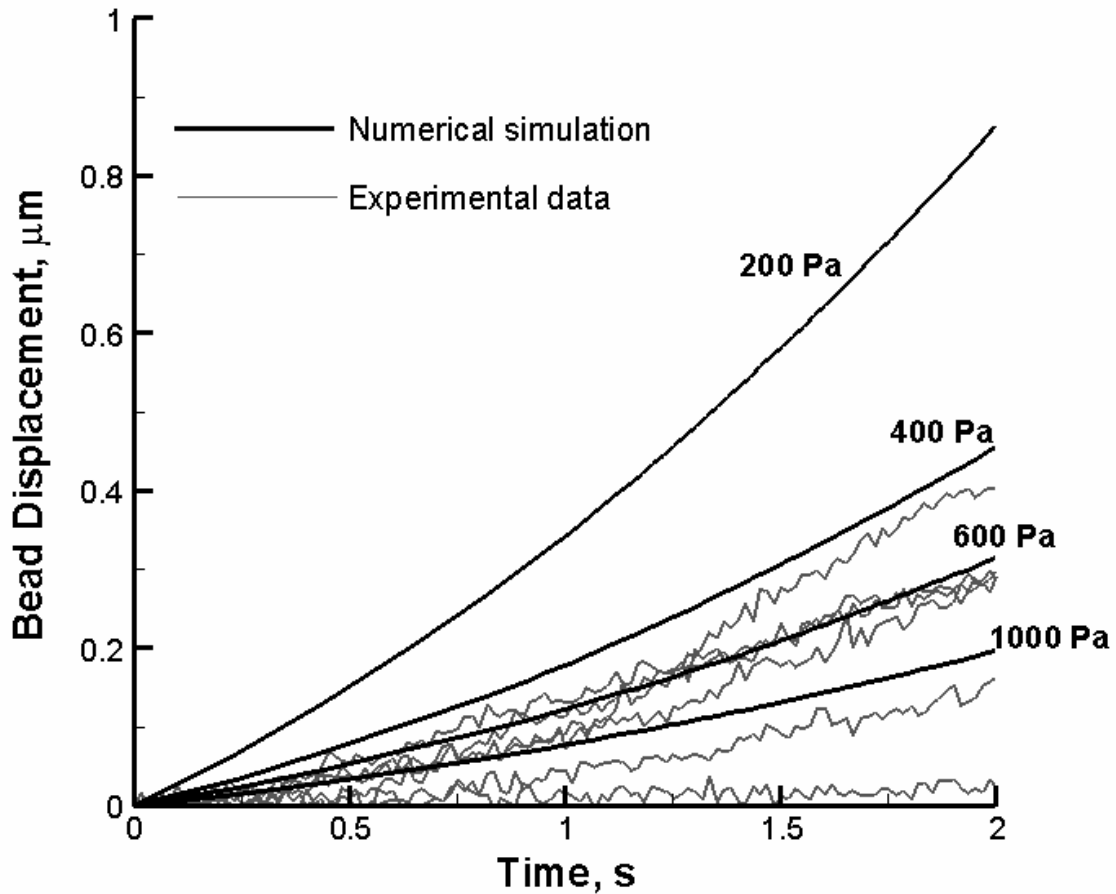


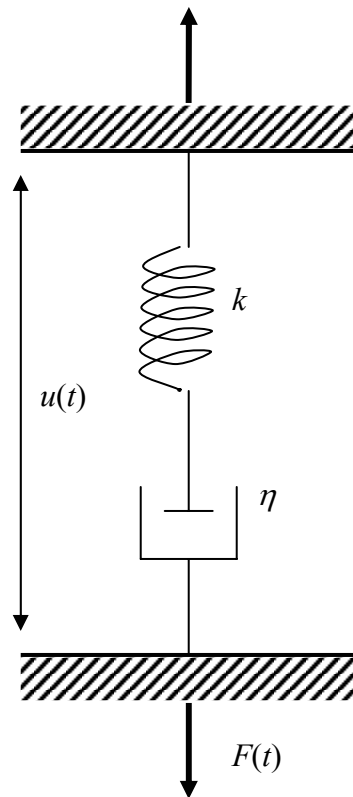
Fig. 1. (Karcher et al., 2003)

The “Numerical Simulation” curves are obtained by using a Maxwell model to describe the cell cytoskeleton and membrane.

Making the gross simplification that we can model the cell’s response using a simple, lumped-parameter Maxwell model (Fig. 2), answer the following questions:

(a) Find the linear, constant coefficient ordinary differential equation relating force ( $F$ ) and strain or deformation ( $u$ ) in terms of the spring and dashpot parameters  $k$  and  $\eta$  identified in Fig. 2.

Fig. 2



(b) Using the model in (a), find values of  $k$  and  $\eta$  that *approximately* fit the experimental data in Fig. 1. (Obviously the experimental results fall in a wide range, but find values that fall roughly in the middle of the data.)

(c) What is your assessment of the fit between the experimental data and the Maxwell model? Do you think that another type of model (with a different arrangement of lumped elements) would fit better? Why or why not?

## Problem 2 3-Element Model for Viscoelastic Materials

(a) In class, we stated that the 3-element model for the standard linear viscoelastic solid (Fig. 1a) could be described by an ordinary differential equation of the form:

$$F + \alpha \frac{dF}{dt} = k_1 u + \beta \frac{du}{dt}$$

Show that this form is correct, and find the constant  $\alpha$  and  $\beta$  in terms of the element values  $k_1$ ,  $k_2$ , and  $\eta$ .

(b) The differential equation given above is useful for describing creep and stress relaxation of some viscoelastic materials. For the case of cyclic (sinusoidal) loading it is more convenient to describe the frequency dependence of the material stress-strain behavior using a “complex modulus”. In class, we showed that the complex modulus has the form:

$$\hat{G} = G' + iG''$$

where  $\hat{G} = (k_1 + i\omega\beta)/(1 + i\omega\alpha)$ . Find the real ( $G'$ ) and imaginary ( $G''$ ) parts of the complex modulus in terms of  $\alpha$ ,  $\beta$  and the radian frequency  $\omega$ .

(c) Show that  $G'(\omega)$  and  $G''(\omega)$  have the frequency dependence shown qualitatively in Fig. 1b by reasoning the low and high frequency limits. Find the constants A and B in terms of the element values  $k_1$  and  $k_2$  based on physical (and/or mathematical) arguments.

Fig. 1(a)

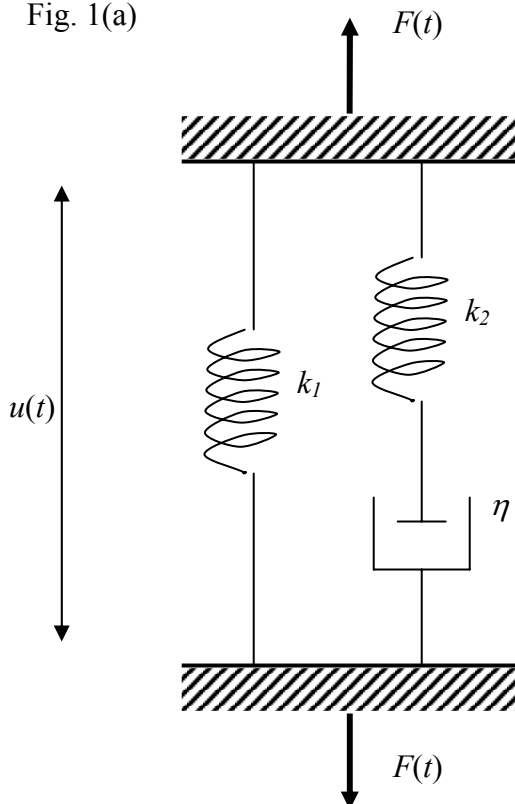
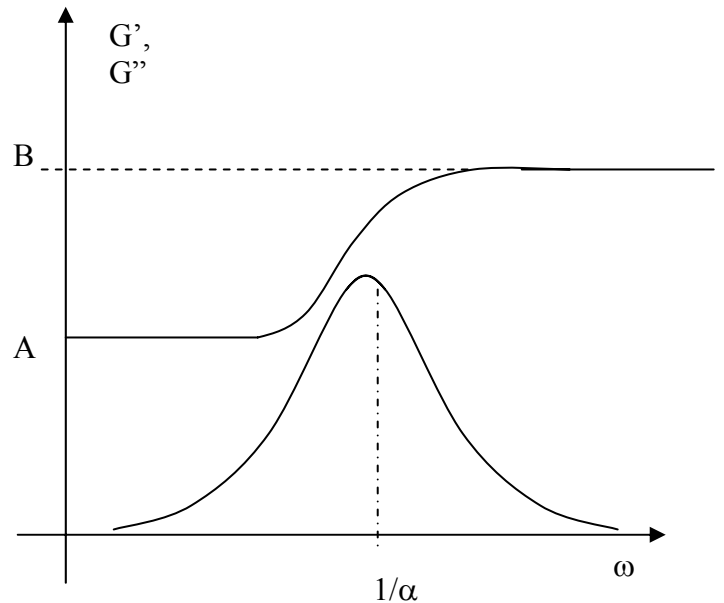


Fig. 1(b)



### Problem 3 3-Element Model for a Viscoelastic Fluid

The 2-dashpot and spring model of Fig. 1 below is a simple representation of a polymeric fluid that exhibits viscoelasticity. Important examples include synovial (joint) fluid and the jelly-like vitreous humor of the eye. Both these bio-fluids contain entangled networks of the long-chain GAG molecule hyaluronan (or hyaluronic acid), which gives the fluid its viscoelastic properties.

(a) It can be shown that the spring-dashpot model of Fig. 1 is described by the ordinary, linear differential equation relating force  $F$  and deformation (or strain)  $u$ :

$$F + \alpha \frac{dF}{dt} = \beta \frac{du}{dt} + \gamma \frac{d^2u}{dt^2}$$

where  $\alpha$ ,  $\beta$ , and  $\gamma$  are constants. For the case of cyclic (sinusoidal) loading, the stress-strain behavior of the material is described by the complex modulus:

$$\hat{G} = G' + iG''$$

Find the real ( $G'$ ) and imaginary ( $G''$ ) parts of the complex modulus in terms of  $\alpha$ ,  $\beta$ ,  $\gamma$  and the radian frequency  $\omega$ . (You need not relate  $\alpha$ ,  $\beta$ ,  $\gamma$  to the spring or dashpot constants.)

(b) State any physical constraints that must be satisfied by the values of the constant  $\alpha$ ,  $\beta$ , and  $\gamma$ .

(c) Sketch (qualitatively) the dependence of  $G'(\omega)$  and  $G''(\omega)$  on the frequency  $\omega$ . Provide a physical argument to justify the low and high frequency limits of your sketch.

Fig. 1

