9.07 Introduction to Statistics for Brain and Cognitive Sciences
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Homework Assignment 4
October 12, 2016
Due October 19, 2016 at 5:00 PM

1. The joint probability mass function of two discrete random variables X and Y is given by

$$f(x) = \begin{cases} c(4x+3y) & 0 \le x \le 4 \quad 0 \le y \le 2 \\ 0 & \textit{otherwise} \end{cases}$$

where x and y are all integer values.

A. Find the value of c to make $f(x,y)$ a well-defined joint pmf. (Hint: Make a table of the values of x and y and the associated values of $f(x,y)$.)

B. Find $\Pr(X \le 1, Y \ge 1)$

C. Find the conditional pmf of X given $Y = 1$.

D. Find the $E[X \mid Y = 1]$.

E. Find the $Var[X \mid Y = 1]$.

2. If X and Y have the joint density function

$$f(x,y) = \begin{cases} \frac{3}{4} + xy & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

A. Find $f(y \mid x)$

B. Compute $\Pr(Y > \frac{1}{2} \mid X = \frac{1}{2})$

C. Compute $E[Y \mid X]$.

D. Compute $Var[Y \mid X]$.

3. Suppose that X and Y have the joint probability density function

$$f(x,y) = xe^{-x(y+1)} \qquad 0 < x < \infty \qquad 0 < y < \infty$$

A. Find the marginal densities of X and Y. Are X and Y independent?

B. Find the conditional densities of X and Y.

4. The objective of this question is to show that the normalization constant for the Gaussian distribution is $(2\pi\sigma^2)^{\frac{1}{2}}$. We take 4 steps to demonstrate that

$$J = \int_{-\infty}^{\infty} \exp\{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2\}dx = (2\pi\sigma^2)^{\frac{1}{2}}$$

A. Make the change of variable $u = \dfrac{x-\mu}{\sigma}$ and consider

$$I = \int_{-\infty}^{\infty} \exp\{-\frac{1}{2}u^2\}du$$

B. Compute I^2

$$I^2 = I \times I = \int_{-\infty}^{\infty} \exp\{-\frac{1}{2}u^2\}du \int_{-\infty}^{\infty} \exp\{-\frac{1}{2}v^2\}dv$$

Make the change of variables $u = r\cos\theta \quad v = r\sin\theta$ and show that

$$I^2 = \int_0^{\infty}\int_0^{2\pi} r\exp\{-\frac{1}{2}r^2\}d\theta dr$$

(Hint: Draw a picture of the transformation $u = r\cos\theta \quad v = r\sin\theta$. Use the change-of-variables formula on page 3 of the **Addendum to Lecture 3**)

C. Show that $I^2 = 2\pi$

D. The result now follows by showing that $I = (2\pi)^{\frac{1}{2}}$ and $J = I\sigma$.

5. Assume X, Y are bivariate Gaussian random variables with parameters $\mu_x, \mu_y, \sigma_x^2, \sigma_y^2$ and ρ. The marginal distribution of Y is Gaussian with mean μ_y and variance σ_y^2. Show that Y given X is Gaussian with

$$E(Y \mid X) = \mu_y + \rho \frac{\sigma_y}{\sigma_x}(x - \mu_x) \tag{1}$$

$$\mathrm{Var}(Y \mid X) = \sigma_y^2(1 - \rho^2) \tag{2}$$

A. Make the change of variables $U = \dfrac{X - \mu_x}{\sigma_x}$ and $V = \dfrac{Y - \mu_y}{\sigma_y}$ and show that U and V are bivariate Gaussian with $E(U) = E(V) = 0, \mathrm{Var}(U) = \mathrm{Var}(V) = 1$ and $\rho = \rho.$

B. Use the definition of the conditional distribution of V given U to show that it is a Gaussian distribution with

$$E(V \mid U = u) = \rho u$$

$$\mathrm{Var}(V \mid U = u) = 1 - \rho^2$$

(Hint: You will need to complete the square in V.)

C. Equations 1 and 2 follow by making the change of variable $X = \sigma_x u + \mu_x \quad Y = \sigma_y V + \mu_y.$
(Hint: Remember only Y is a random variable. Why is this important?)

6. Let X and Y be independent standard Gaussian random variables. Find the joint density of $U = Y + X$ and $V = Y + X.$ (Hint: Use **Example 5.7**).

7. The buses on the north-line and the bus on the east-west line both stop at the same stop. Suppose that the waiting time T_1 for the bus on the north-south line is an exponential random variable with parameter α and suppose that that the waiting time T_2 for the bus on the east-west line is an exponential random variable with parameter β.

A. Compute the probability that the bus on the north-south line arrives first. That is, $\Pr(T_2 > T_1)$.

B. Compute $\Pr(T_2 > 2T_1)$. What is the interpretation of this probability?