Problem Set 8 (due Thurs May 6)
Models of Associative Memory

April 29, 2004

1. Capacity of the Hopfield Model.

   Consider the Hopfield model, using a sequential update, given by

$$s_i = \text{sign}(\sum_j W_{ij} s_j)$$

$$W_{ij} = \begin{cases} \frac{1}{N}\sum_\mu \xi_i^\mu \xi_j^\mu & i \neq j \\ 0 & i = j \end{cases}$$

   where $s \in \{-1, 1\}^N$. (Sequential update means that the neurons are updated one at a time, typically in random order.)

   One definition of the capacity of the Hopfield model is the number of patterns that can be stored where some small fraction ($P_{err} \leq 0.01$) of the bits are corrupted. Using this definition, the capacity of the original Hopfield model is approximately $0.14N$ for large N, where N is the number of units in the network. In this problem, we will validate this capacity using a simple MATLAB simulation, and then use our simulation to compare the capacities of original Hopfield model with the capacities of a network storing sparse patterns using $\{0, 1\}$ units.

   (a) Construct P random $\{-1, 1\}$ patterns, $\xi^1, ..., \xi^P$, each of size N. Find W using the prescription given above.

   We investigate the capacity by checking if each of the stored patterns are actually steady states of the system. The weight update from a stored pattern ξ^ν can be written as:

$$s_i = \text{sign}(\xi_i^\nu + \frac{1}{N}\sum_{\mu\neq\nu}\sum_{j\neq i} \xi_i^\mu \xi_j^\mu \xi_j^\nu).$$

   We would like s_i to equal ξ_i^ν, but our steady state could be corrupted by the zero-mean crosstalk term. To visualize this in MATLAB, collect the terms $\sum_j W_{ij}\xi_j^u$ for all i and all μ and make a histogram of the results. To get a nice plot, use $N = 1000$ and 50 bins instead of MATLAB's default of 10.

   Submit your matlab code and plots for $P = 100, 200$, and 140 (the known capacity for $N \to \infty$). Describe in words how the shape of the histogram changes as we change P, and how this impacts the capacity.

2. Storing binary patterns using the covariance rule.

   In this problem, we will consider a sparse network. This means that instead of the $\{-1, 1\}$ network used in the previous problem, we will use $\{0, 1\}$ units. The patterns that we wish to store are random with each bit having probability f of being a 1. We are interested in the case where f is small.

   The network is defined by the covariance rule:

$$W_{ij} = \begin{cases} \frac{1}{Nf(1-f)}\sum_\mu (\xi_i^\mu - f)(\xi_j^\mu - f) & i \neq j \\ 0 & i = j \end{cases}$$

with the discrete dynamics:

$$x_i = H(\sum_j W_{ij} x_j - \theta_i)$$

where H is the Heaviside function: $H(u) = 1$ if $u > 0$, otherwise $H(u) = 0$.

(a) Show that for large N and small f the sum $\sum_j W_{ij}\xi_j^\nu$ can be separated into ξ_i^ν and a crosstalk term.

(b) Show that this crosstalk term has zero mean.

(c) Construct P random $\{0, 1\}$ patterns, each of size N, using f as the probability of a 1 bit. Plot the histogram of $\sum_j W_{ij}\xi_j^\mu$ as in part a. Experiment with P to estimate the capacity for $N = 1000$ and $f = 0.05$.

(d) According to your simulations, what value of the threshold θ_i maximizes the capacity?

(e) One published result estimates the capacity of the sparse network as $P = \frac{N}{2f|\log(f)|}$. How well does this quantity compare to your results (test this by varying N and f)?

3. Storing binary patterns using another rule.

As in the previous problem, we will consider a network with $\{0, 1\}$ units but with a different rule:

$$W_{ij} = \begin{cases} \frac{1}{Nf}\sum_\mu(\xi_i^\mu\xi_j^\mu - f^2) & i \neq j \\ 0 & i = j \end{cases}$$

with the discrete dynamics:

$$x_i = H(\sum_j W_{ij} x_j - \theta_i).$$

(a) Repeat (a)-(d) from the previous problem using this network.

(b) Extra credit: Derive an expression for the capacity P in terms of N and f.

4. Compare the capacities of the 3 networks considered.