# Introduction to Neural Computation

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MIT BCS 9.40 — 2018

Lecture 10 - Time Series

# Spatial receptive fields

- How do we represent receptive fields mathematically?

Linear-Nonlinear Model (LN Model)

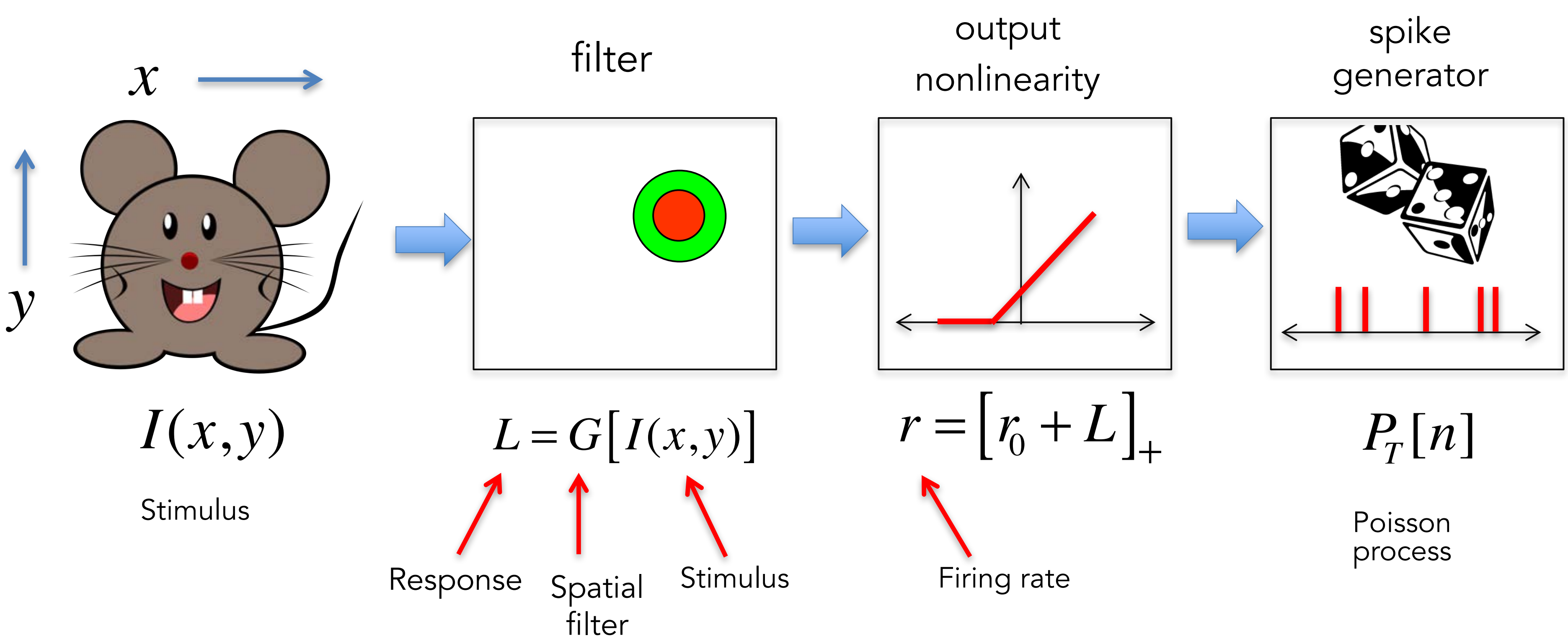



Image of mouse in public domain.

# Spatial receptive fields

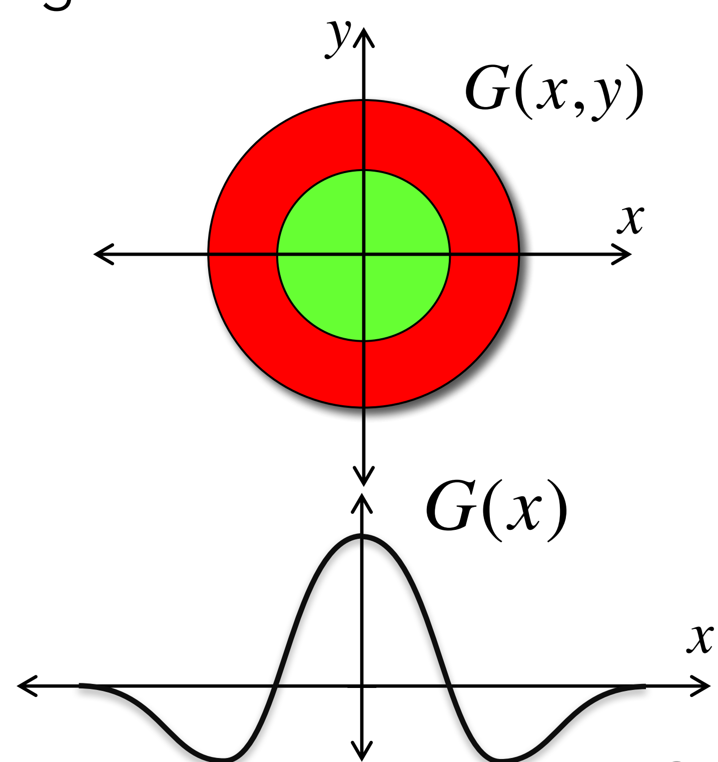
- How do we represent receptive fields mathematically?

We are going to consider the simplest case in which the response of a neuron is given by a linear filter acting on the stimulus.

$$r = r_0 + \iint G(x,y) I(x,y)\, dx\, dy$$

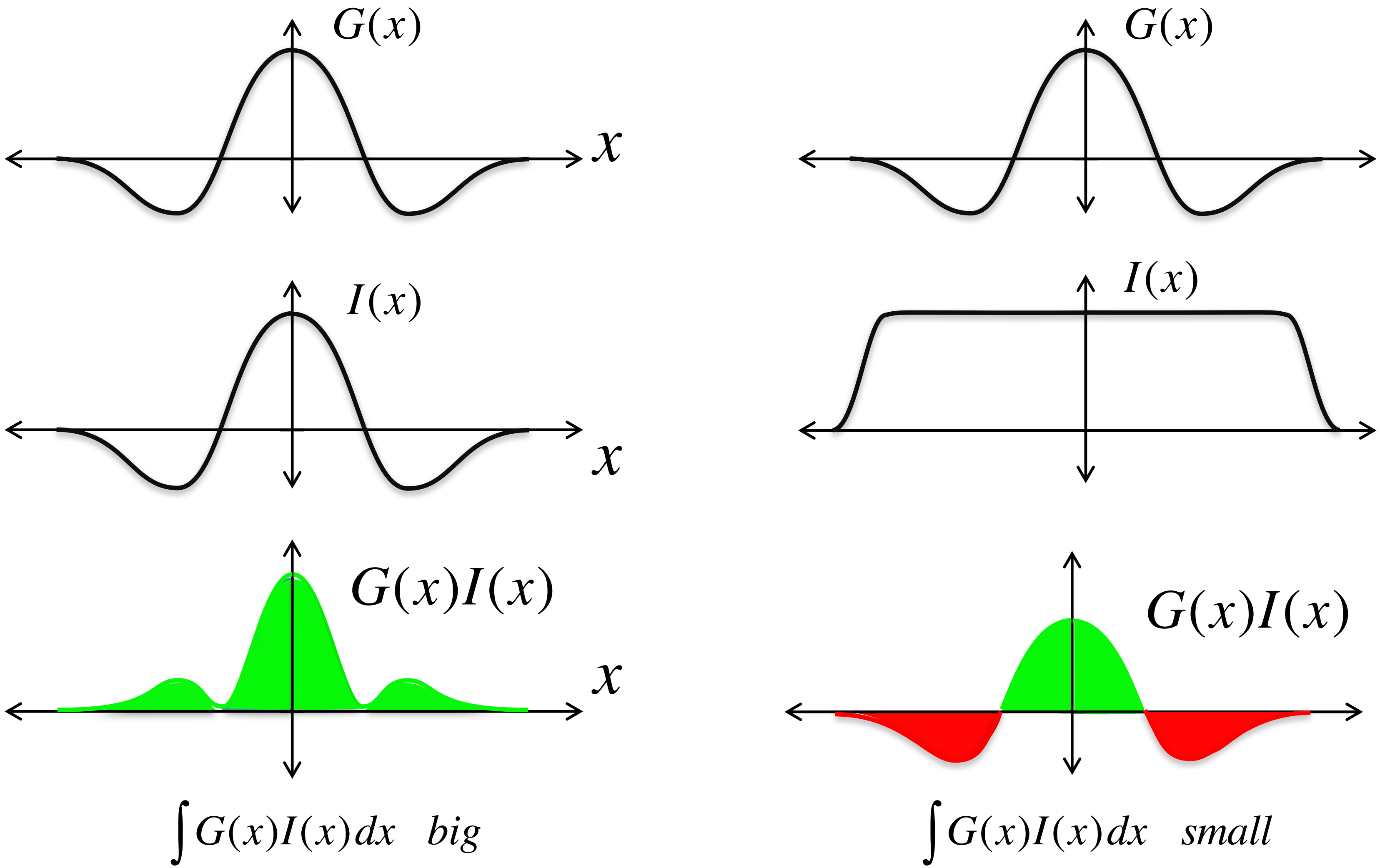
Let's look at this in one dimension

$$r = r_0 + \int G(x) I(x)\, dx$$



# Spatial receptive fields

- How do we represent receptive fields mathematically?



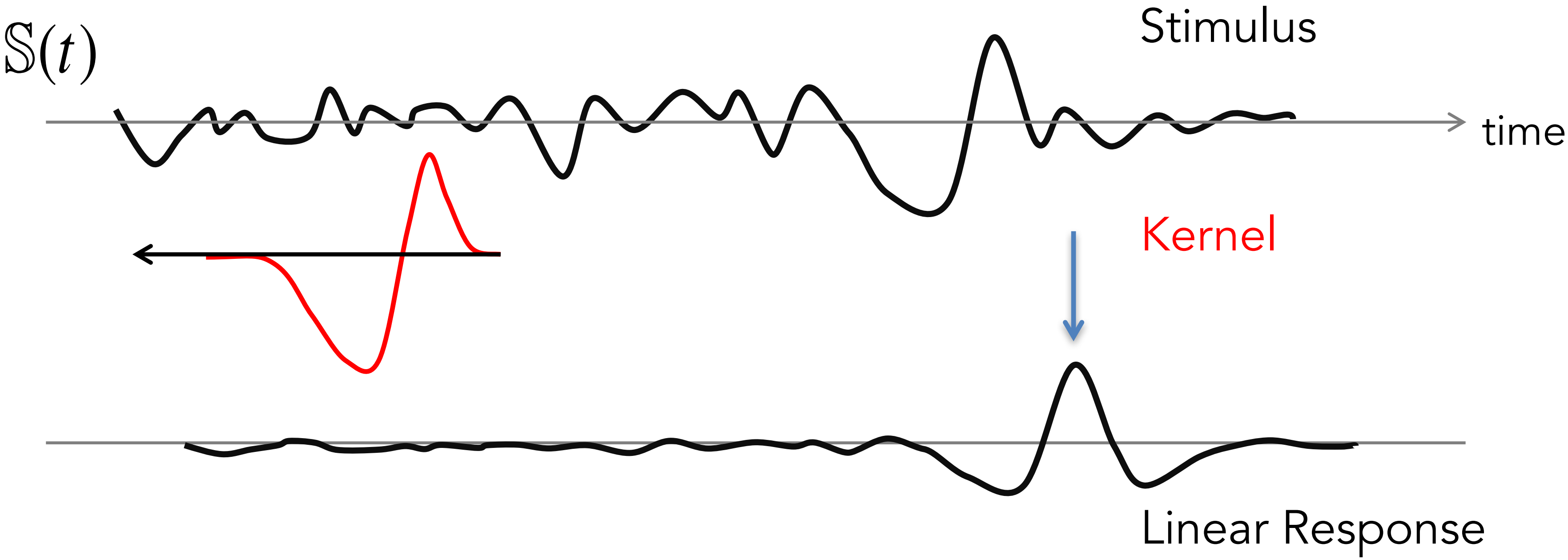
# Temporal receptive fields

- We can also think of the response of a neuron as some function of the temporal variations in the stimulus.

$$r(t) = r_0 + D\left[\mathbb{S}(t)\right]$$

# Temporal receptive fields

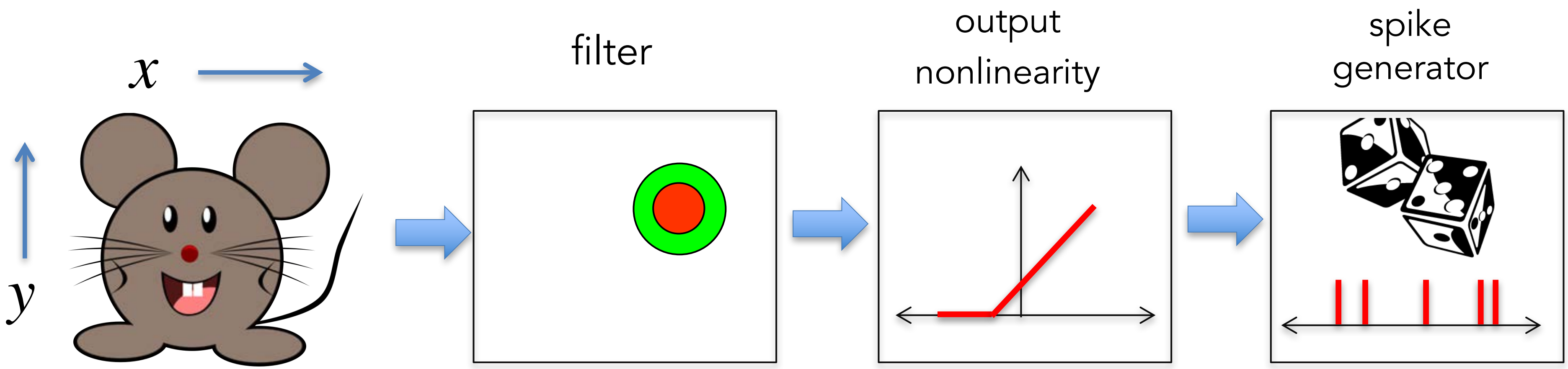
- We can think of ‘overlap’ in the time domain! That there is a particular ‘temporal profile’ of a stimulus that makes a neuron spike.



# Spatio-temporal receptive fields

- How do we represent receptive fields mathematically?

Combine neural responses into a single kernel that captures both spatial and temporal sensitivity.




$I(x,y,t)$ — Stimulus

$L(t) = K\left[I(x,y,t)\right]$ — Response; Spatio-temporal filter; Stimulus

$r(t) = \left[r_0 + L(t)\right]_+$ — Firing rate

$P_T[n]$ — Poisson process

Image of mouse in public domain.

# Learning objectives for Lecture 10

- Spike trains are probabilistic (Poisson Process)
- Be able to use measures of spike train variability
  - Fano Factor
  - Interspike Interval (ISI)
- Understand convolution, cross-correlation, and autocorrelation functions
- Understand the concept of a Fourier series

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# Neuronal responses are variable

- Spike trains are often quite variable. The precise pattern of spikes on each presentation of a stimulus is different.

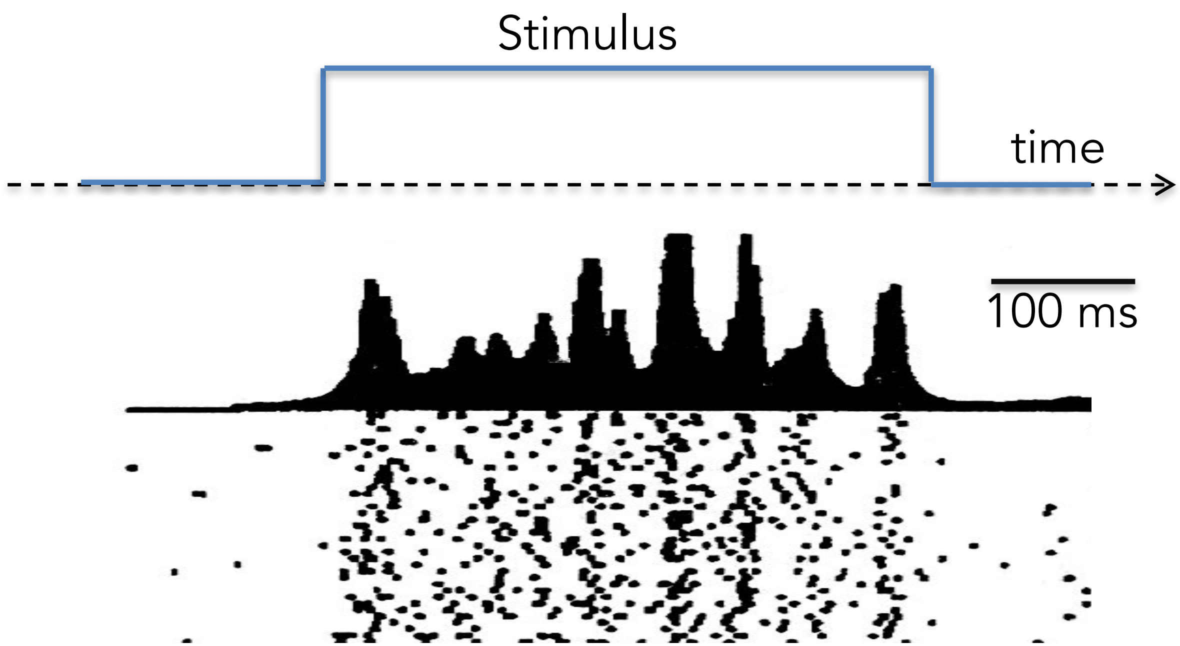
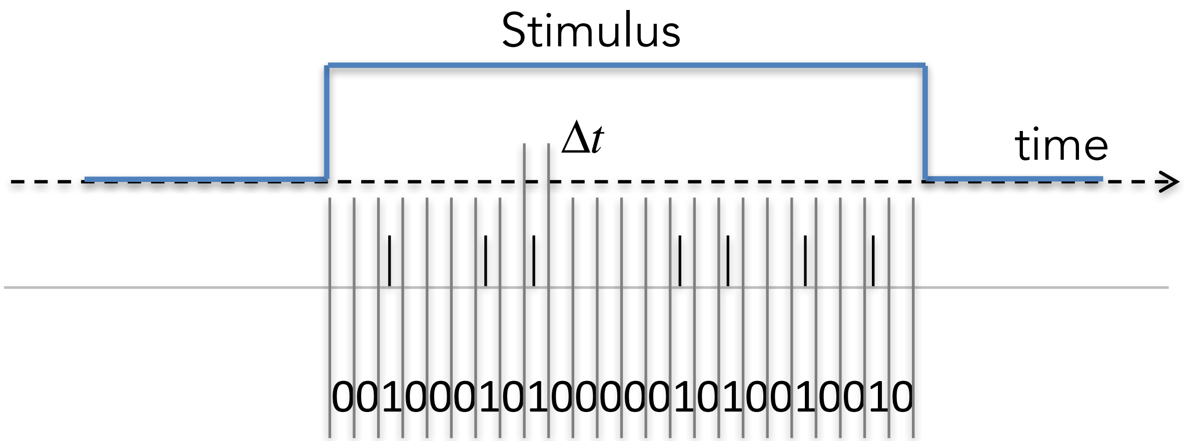



Figure courtesy MIT Press. From Dayan, P. and L. Abbott. *Theoretical Neuroscience: Computational and Mathematical Modeling of Neural Systems*. 2001. Original source: Bair, W. and C. Koch. "Temporal Precision of Spike Trains in Extrastriate Cortex of the Behaving Macaque Monkey." *Neural Computation* 8 no 6 (1996): 1185-1202.

Response of a neuron in area MT of the monkey to the exact same stimulus replayed on each trial.

# Neuronal responses are variable



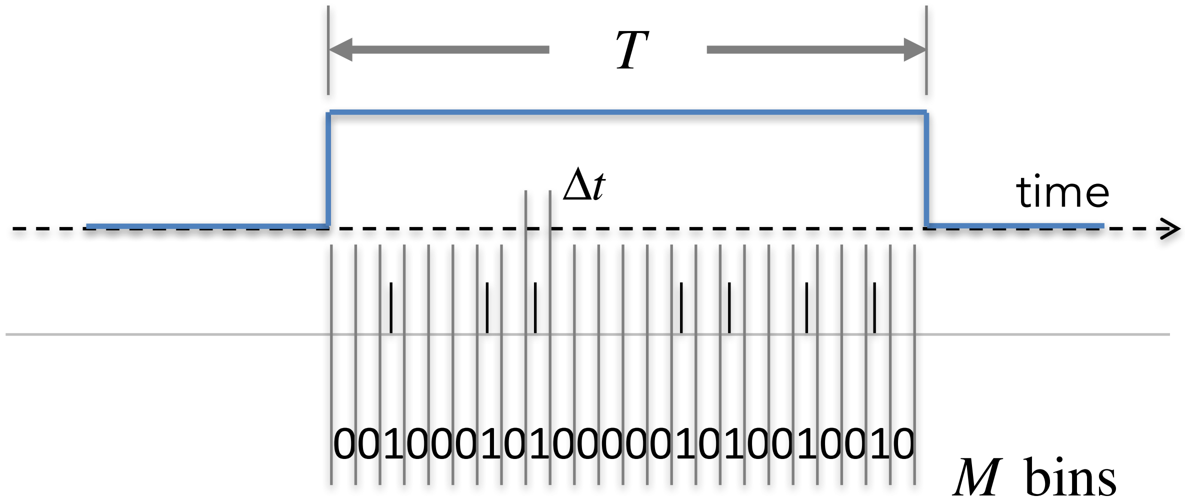

Imagine a random process that produces spikes at an average rate of $\mu$ spikes per second during the stimulus presentation.

Break up the spike train into small time bins of some duration Δt. Each spike is generated independently of other spikes and with equal probability in each bin. then we can write the probability that a spike occurs in any bin as

If Δt is small enough that most of the bins have zero spikes, we can write the probability that a spike occurs in any bin as: $\mu \cdot \Delta t$

The probability that no spike occurs in the bin is: $1-\mu \cdot \Delta t$

# Poisson process




How many spikes land in the interval T ?

What is the probability that n spikes land in the interval T ? $P_T[n]$

This is just the product of three things:

- The probability of having n bins with a spike $= (\mu\,\Delta t)^n$
- The probability of having M-n bins with no spike $= (1-\mu\,\Delta t)^{M-n}$
- The number of different ways to distribution n spikes in M bins $= \dfrac{M!}{(M-n)!n!}$

# Poisson process

What is the probability that n spikes land in the interval T ?

$$P_T[n] = \lim_{\Delta t \to 0} \frac{M!}{(M-n)!n!} (\mu \Delta t)^n (1 - \mu \Delta t)^{M-n}$$

In the limit that: $\Delta t \to 0 \qquad M = \frac{T}{\Delta t} \to \infty$

$$P_T[n] = \frac{(\mu T)^n}{n!} e^{-\mu T}$$

Poisson distribution!

# Poisson distribution

The Poisson Distribution gives us the probability that n spikes land in the interval T

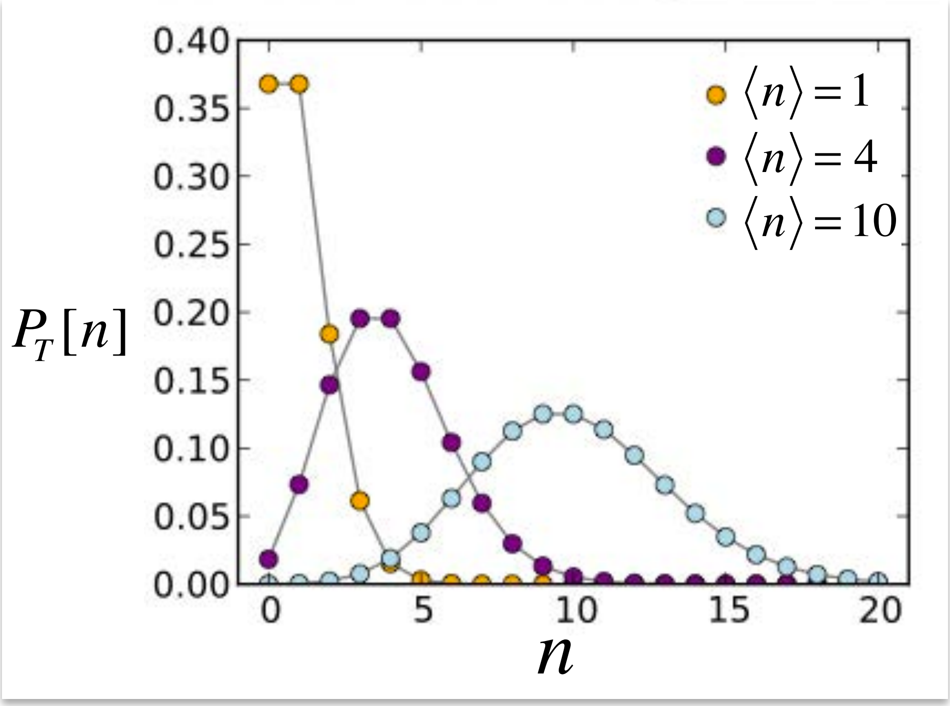
$$P_T[n] = \frac{(\mu T)^n}{n!} e^{-\mu T}$$

Average (expected) number of spikes

$$\langle n \rangle = \sum_{n=0}^{\infty} n P_T[n] = \mu T$$

Thus, $\mu = \frac{\langle n \rangle}{T}$ is also the average

spike rate! (going to use variable r)



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# Spike count variability

What is the variance in the number of spikes that land in the interval T ?

$$P_T[n] = \frac{(\mu T)^n}{n!} e^{-\mu T}$$

Variance in spike count

$$\begin{aligned}\sigma_n^2(T) &= \left\langle \left(n - \langle n \rangle\right)^2 \right\rangle \\ &= \left\langle n^2 \right\rangle - 2\langle n \rangle^2 + \langle n \rangle^2 \\ &= \left\langle n^2 \right\rangle - \langle n \rangle^2\end{aligned}$$

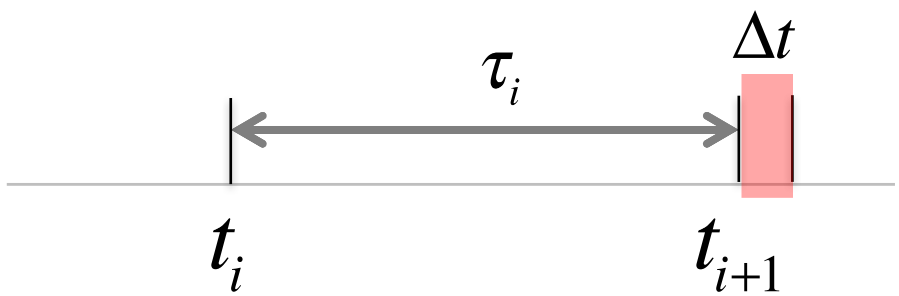
$$\sigma_n^2(T) = \mu T$$

Fano Factor

$$F = \frac{\sigma_n^2(T)}{\langle n \rangle} = 1$$

# Interspike interval (ISI) distribution

What is the distribution of intervals between spikes?




$$\tau_i = t_{i+1} - t_i$$

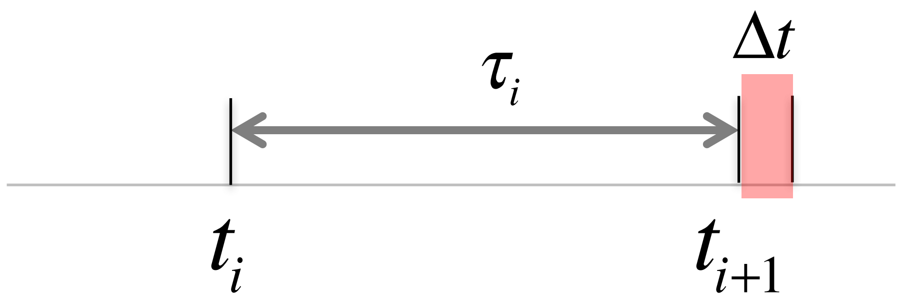
The probability of having the next spike land in the interval between $t_{i+1}$ and $t_{i+1} + \Delta t$ is:

$$P_\tau[n=0] = \frac{(r\tau)^0}{0!} e^{-r\tau} = e^{-r\tau}$$

$$P[\tau \le t_{i+1} - t_i < \tau + \Delta t] = e^{-r\tau}\, r\,\Delta t$$

# Interspike interval (ISI) distribution

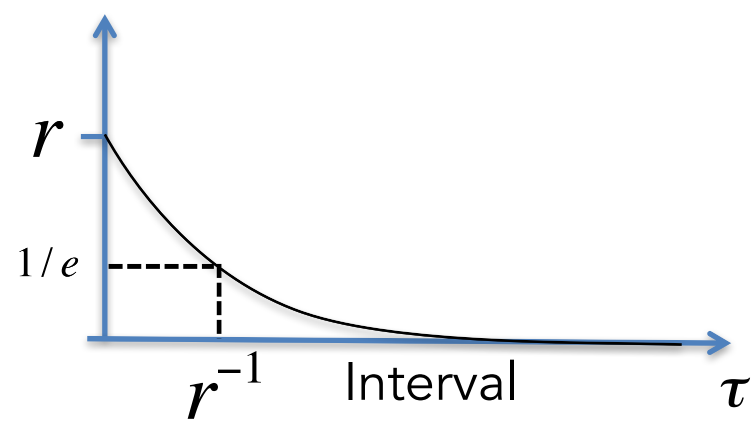
What is the distribution of intervals between spikes?



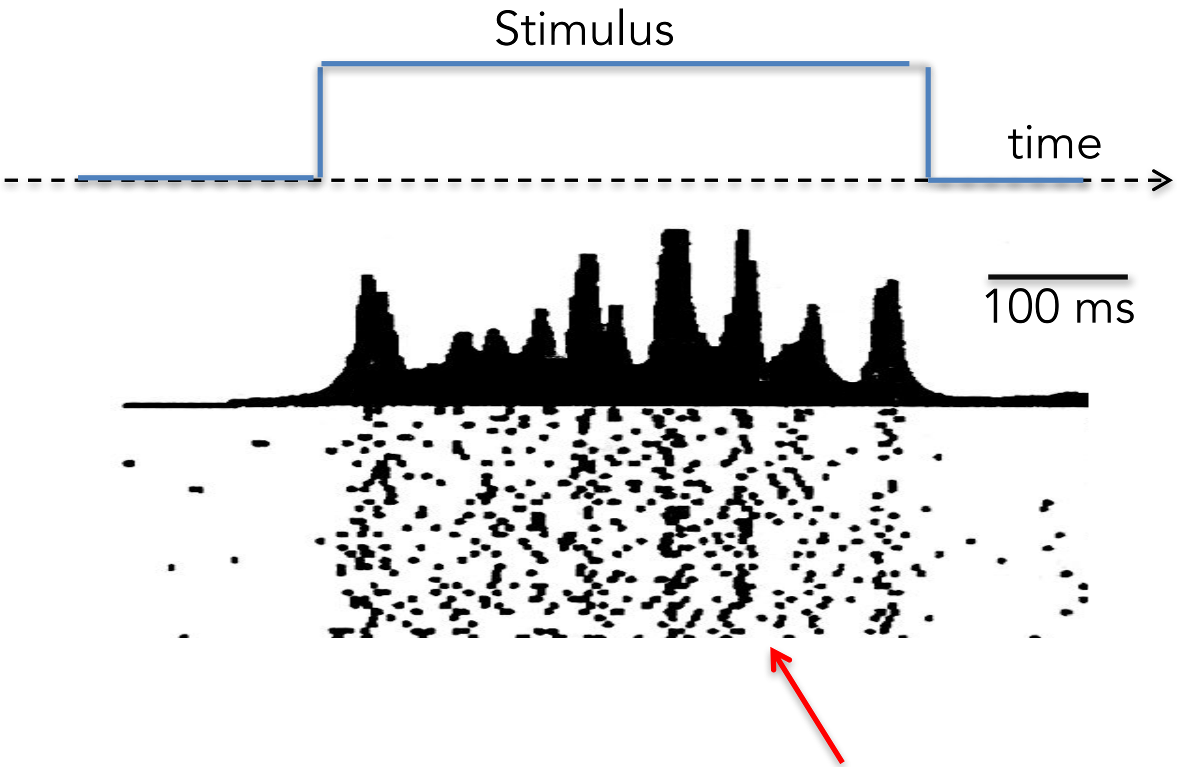

$$\tau_i = t_{i+1} - t_i$$

The probability density (probability per unit time) is just

$$\frac{1}{\Delta t} P[\tau] = r\, e^{-r\tau}$$



# Homogeneous vs inhomogeneous Poisson process




Homogeneous

$rate = \mu$

Inhomogeneous

$rate = \mu(t)$

Annotated figure from Dayan, P. and L. Abbott. *Theoretical Neuroscience: Computational and Mathematical Modeling of Neural Systems*. 2001. Original source: Bair, W. and C. Koch. "Temporal Precision of Spike Trains in Extrastriate Cortex of the Behaving Macaque Monkey." *Neural Computation* 8 no 6 (1996): 1185-1202. 

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# Convolution

- We have discussed the idea of convolution

$$y(t) = \int_{-\infty}^{\infty} d\tau\, G(\tau) x(t-\tau)$$

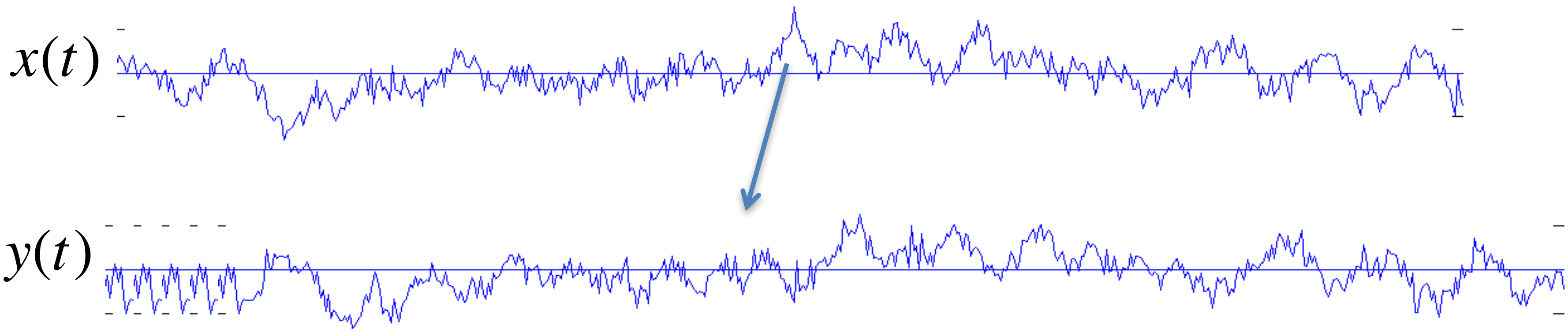
  - To model the response of membrane potential to synaptic input
  - To model the response of neurons to a time-dependent stimulus
  - To implement a low-pass or high-pass filter

- In general, convolution allows us to model the output of a system as a linear filter acting on its input.

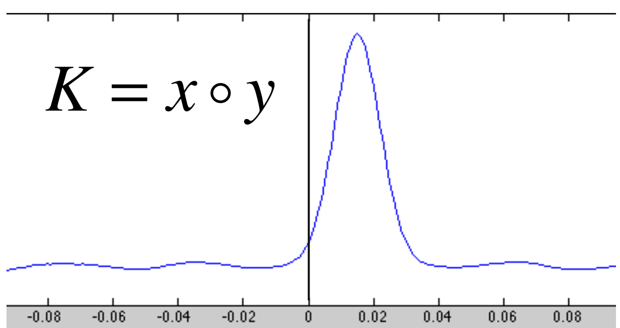
# Cross-correlation function

- A way to examine the temporal relation <u>between</u> signals.

$$K(\tau) = \int_{-\infty}^{\infty} dt\, x(t)\, y(t+\tau)$$




xc=xcorr(ShftNoisyData,NoisyData,Nlags);




Time Lag

# Relation between Convolution and Cross-correlation

- These are mathematically very similar, but are used differently.

**Convolution**

$$y(t) = \int_{-\infty}^{\infty} d\tau\, K(\tau) x(t-\tau)$$

Take input signal $x(t)$ and convolve it with kernel K to get output signal $y(t)$ .

**Cross-correlation**

$$K(\tau) = \int_{-\infty}^{\infty} dt\, x(t) y(t+\tau)$$

Take two signals, $x(t)$ and $y(t)$, and cross-correlate to extract a temporal 'kernel' K.

Think of $x(t)$ and $y(t)$ as long vectors (signals)

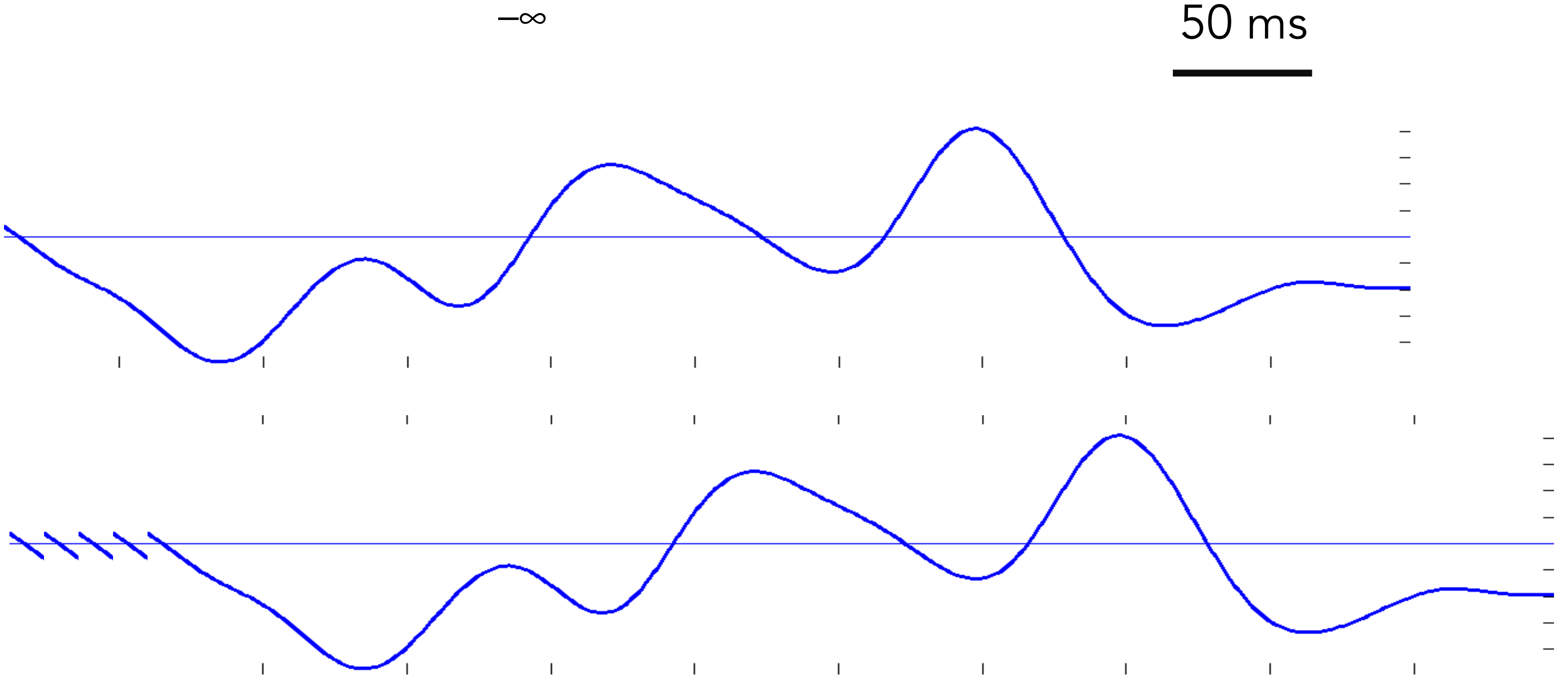
Think of $K(\tau)$ as a short vector (kernel)

Relation to STA

# Autocorrelation

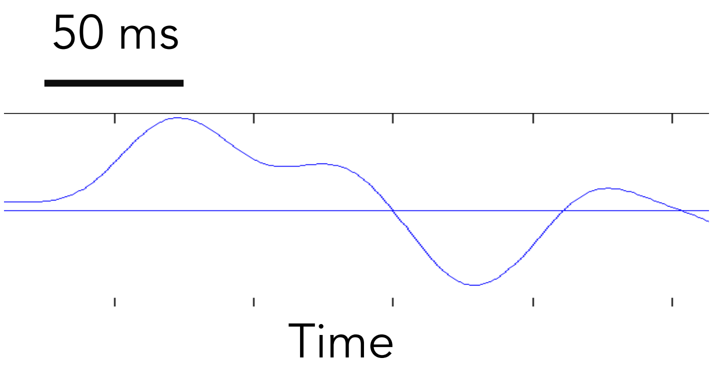
- A way to examine the temporal structure within a signal.

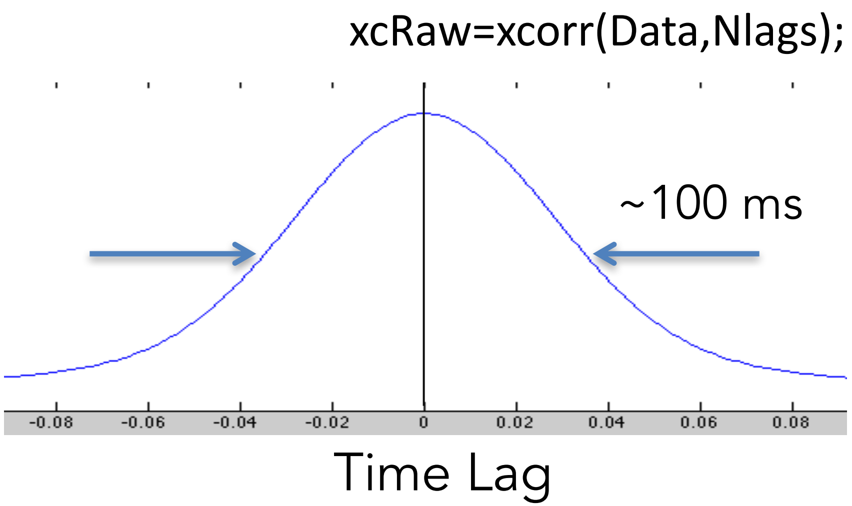
$$K(\tau) = \int_{-\infty}^{\infty} dt\, x(t)x(t+\tau)$$



# Autocorrelation

- A way to examine the temporal structure within a signal.



# Autocorrelation

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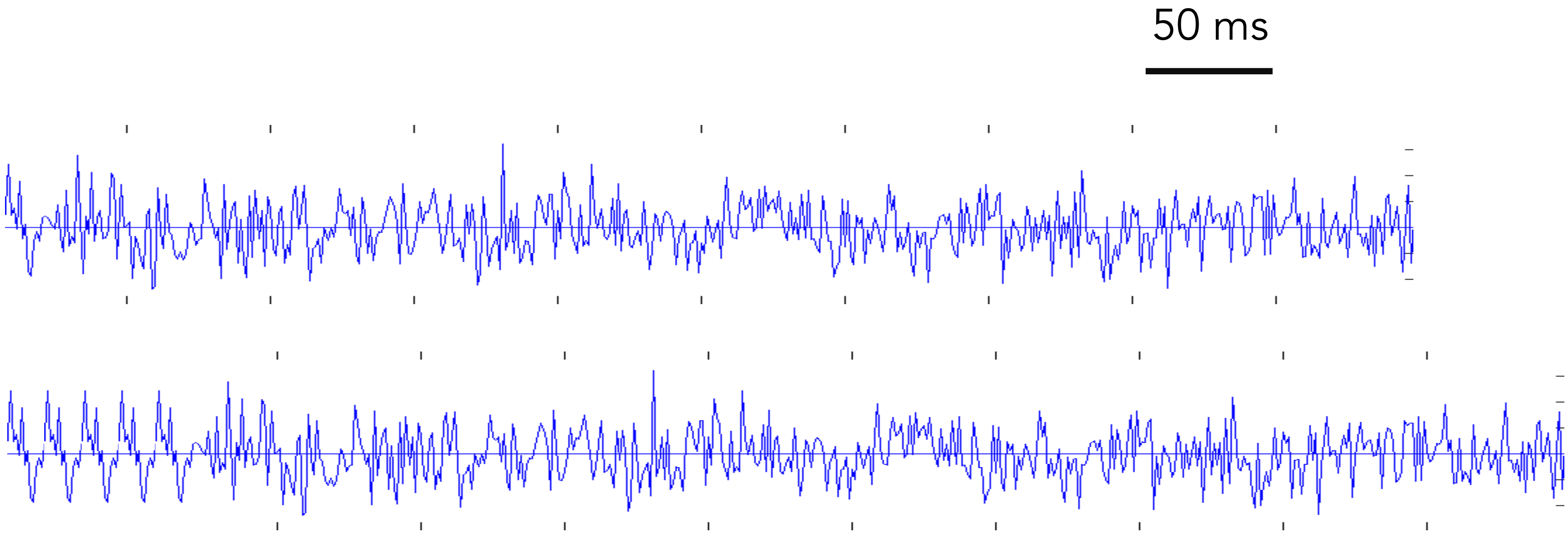
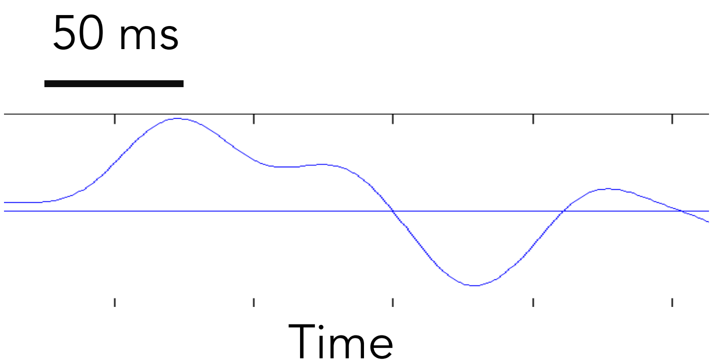
$$K(\tau) = \int_{-\infty}^{\infty} dt\, x(t)x(t+\tau)$$

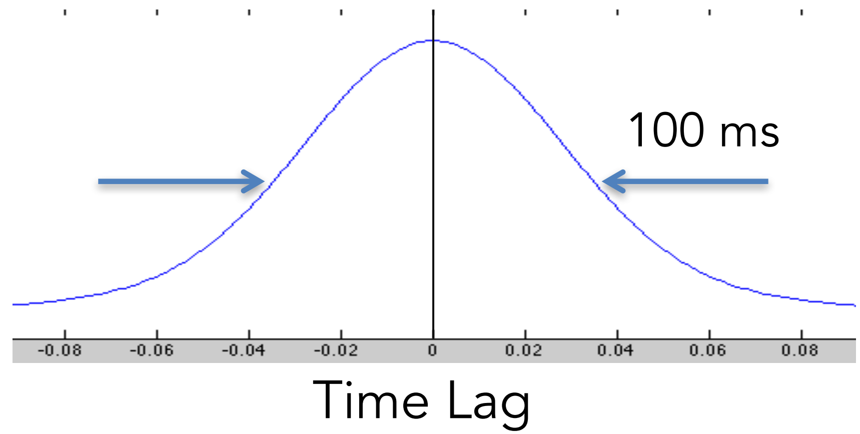


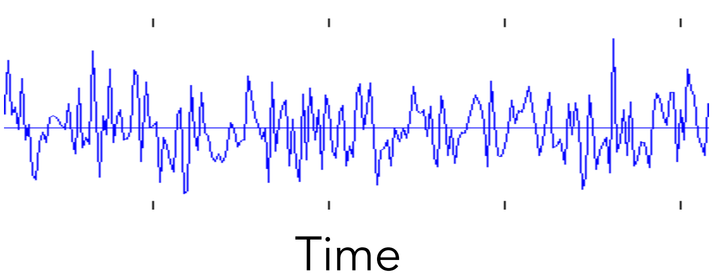
# Autocorrelation

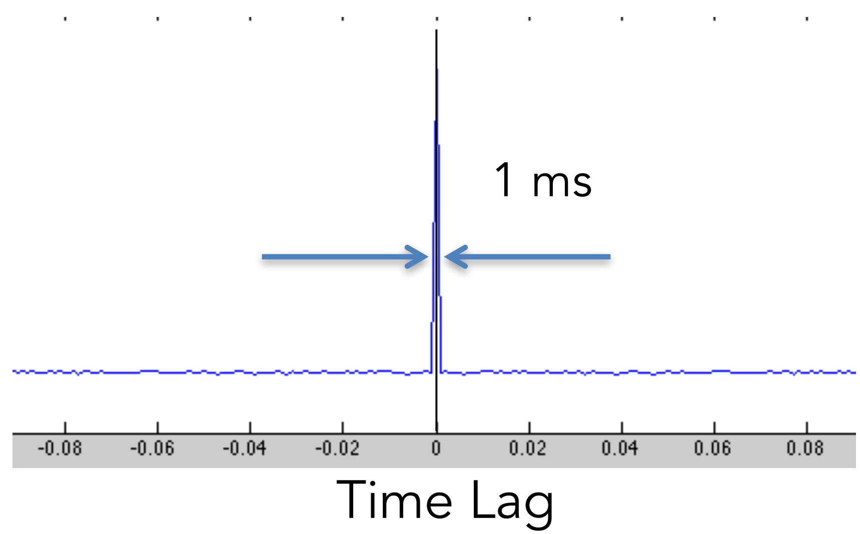
- A way to examine the temporal structure within a signal.

xcRaw=xcorr(Data,Nlags);



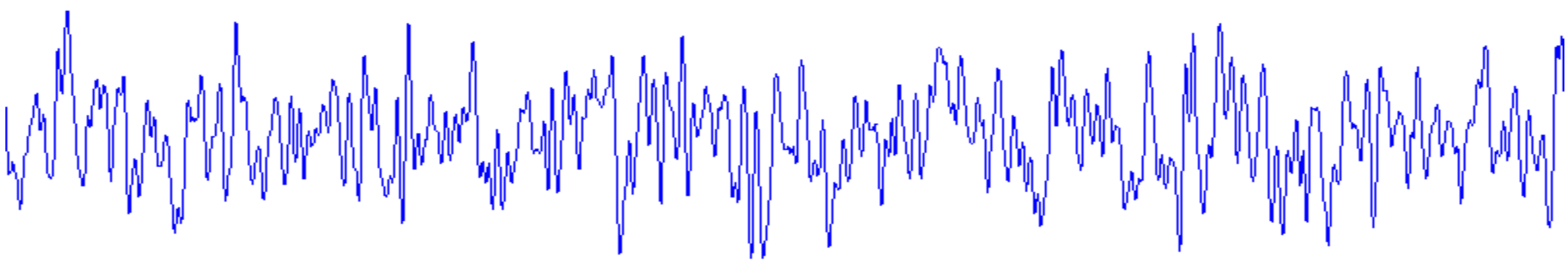



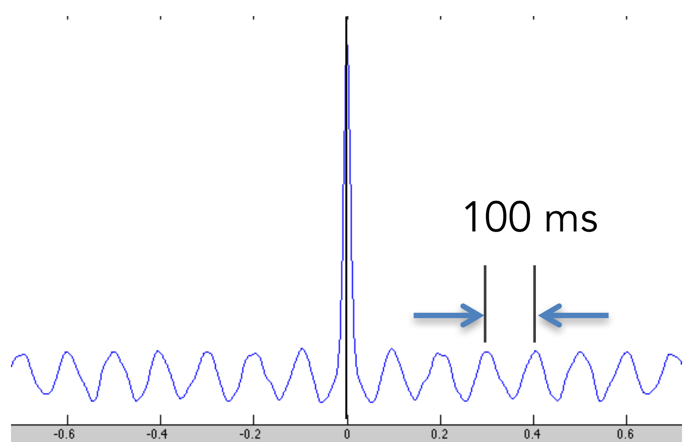



# Autocorrelation

- A way to examine the temporal structure within a signal.

```
Data = randn(1,N)+0.1*cos(2*pi*10*time);
```



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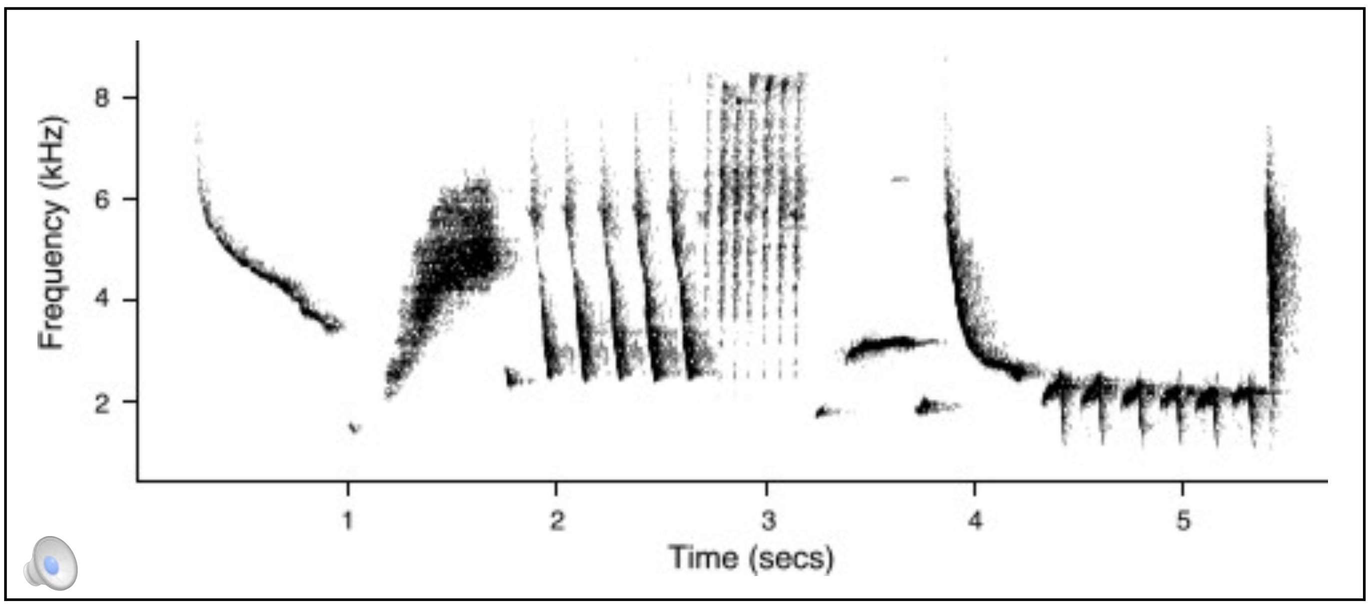
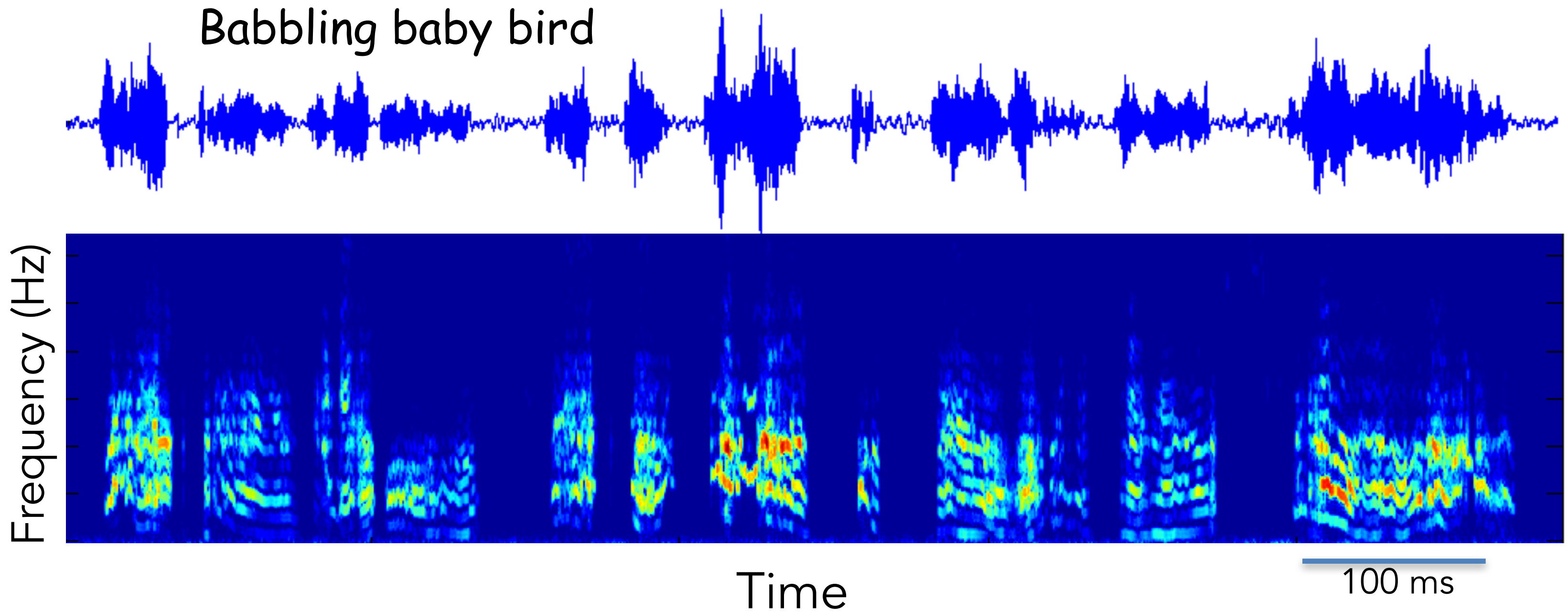
# Spectral Analysis

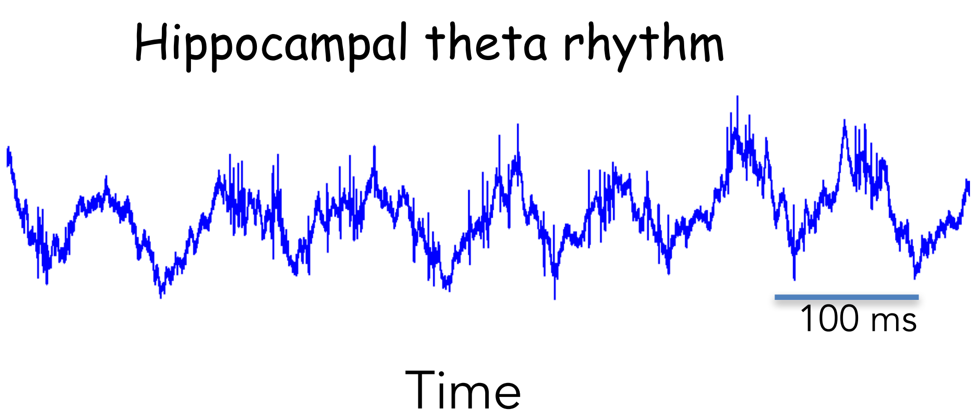
A spectrogram shows how much power there is in a sound at different frequencies and at different times.

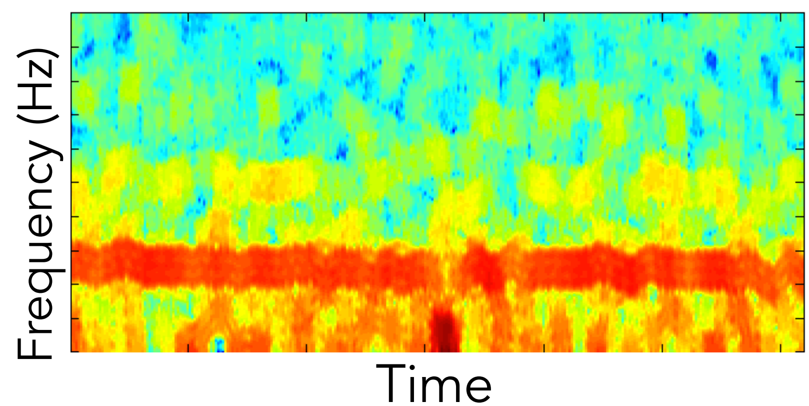
$$S(f,t)$$



# Spectral Analysis

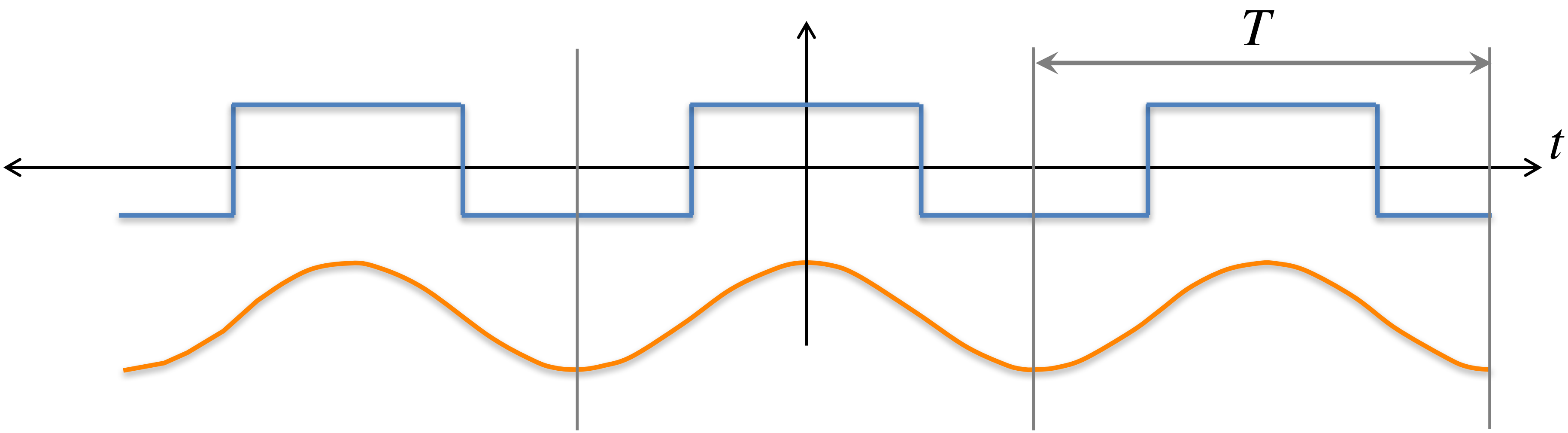


# Fourier Series

- We can express any periodic function of time as sums of sine and cosine functions.
- Let's start with an even function that is periodic with a period T




We could approximate this square wave with a cosine wave of the same period T and amplitude.

$$a_1 \cos(2\pi f_0 t)$$

Oscillation frequency

$$f_0 = \frac{1}{T}$$

Cycles per second (Hz)

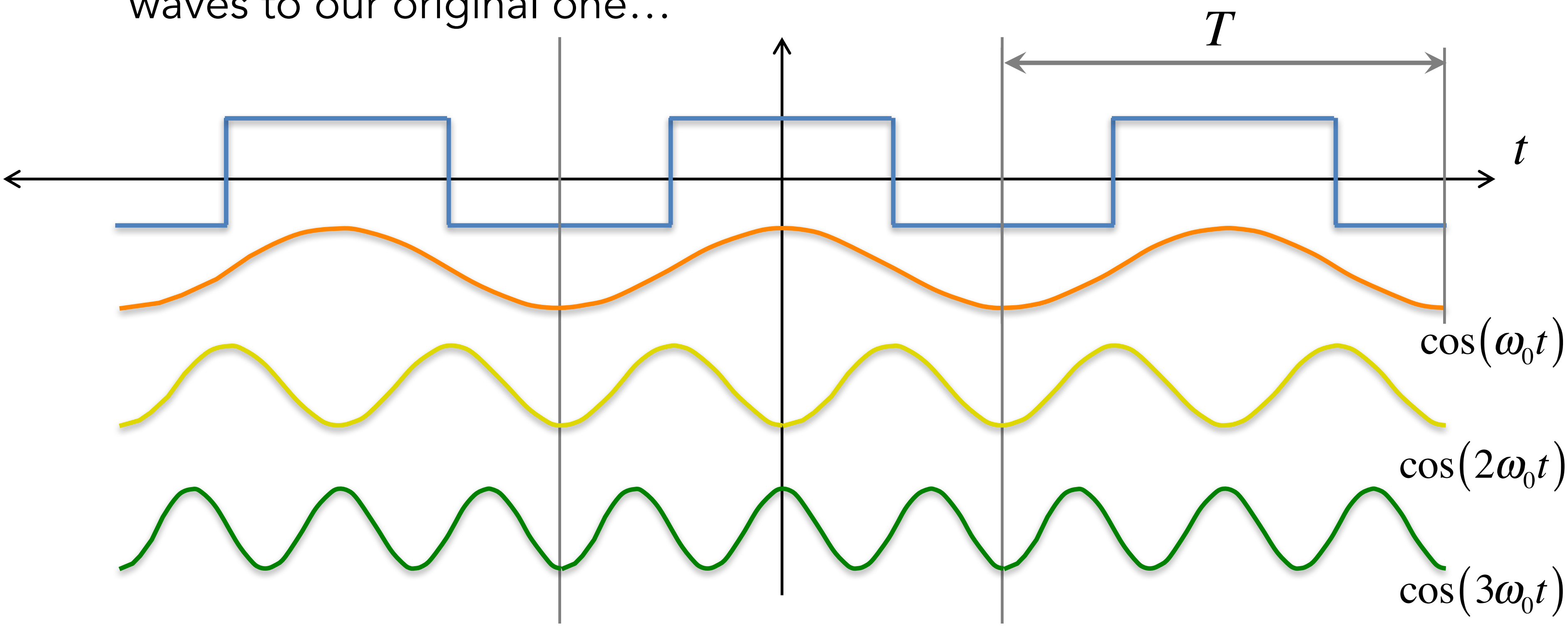
Angular frequency

$$\omega_0 = \frac{2\pi}{T}$$

Radians per second

# Fourier Series

- But we can get a better approximation if we add some more cosine waves to our original one…

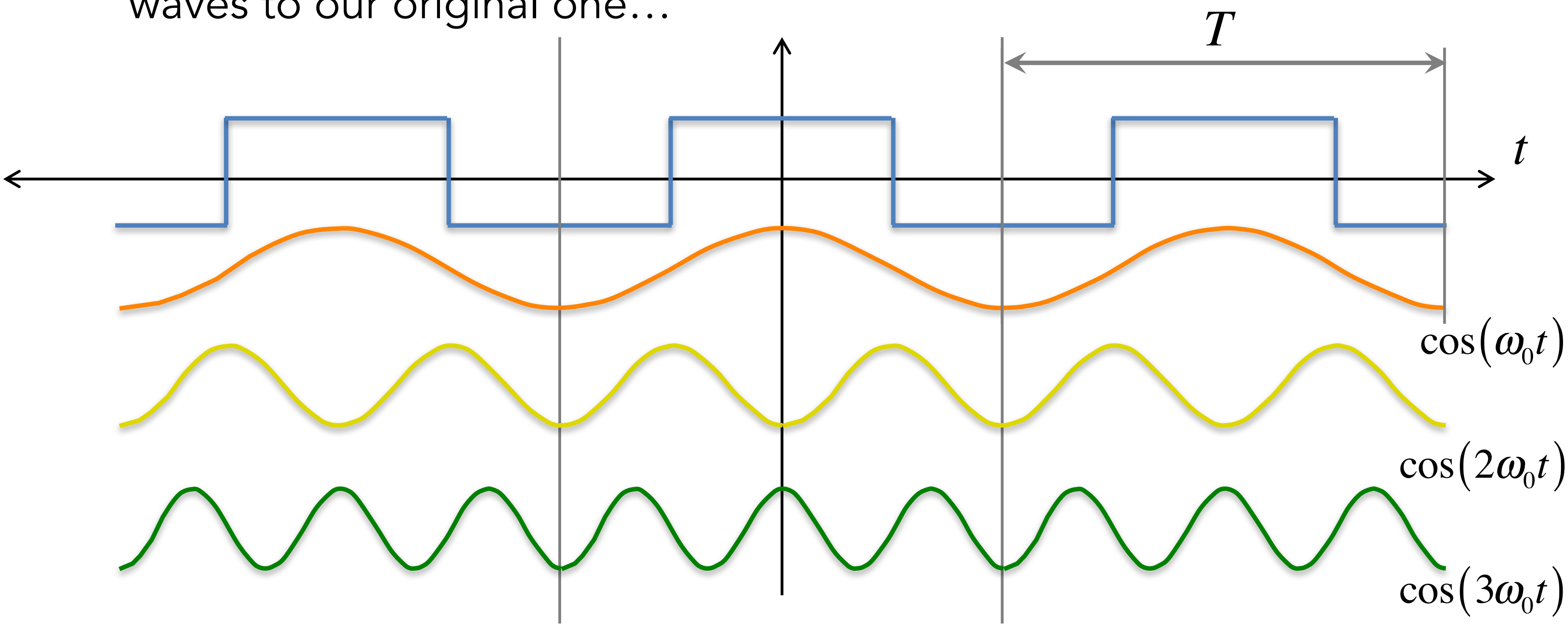



Why can we restrict ourselves to only frequencies that are integer multiples of $\omega_0$ ?

Because only cosines that are integer multiples of $\omega_0$ are periodic with a period T!

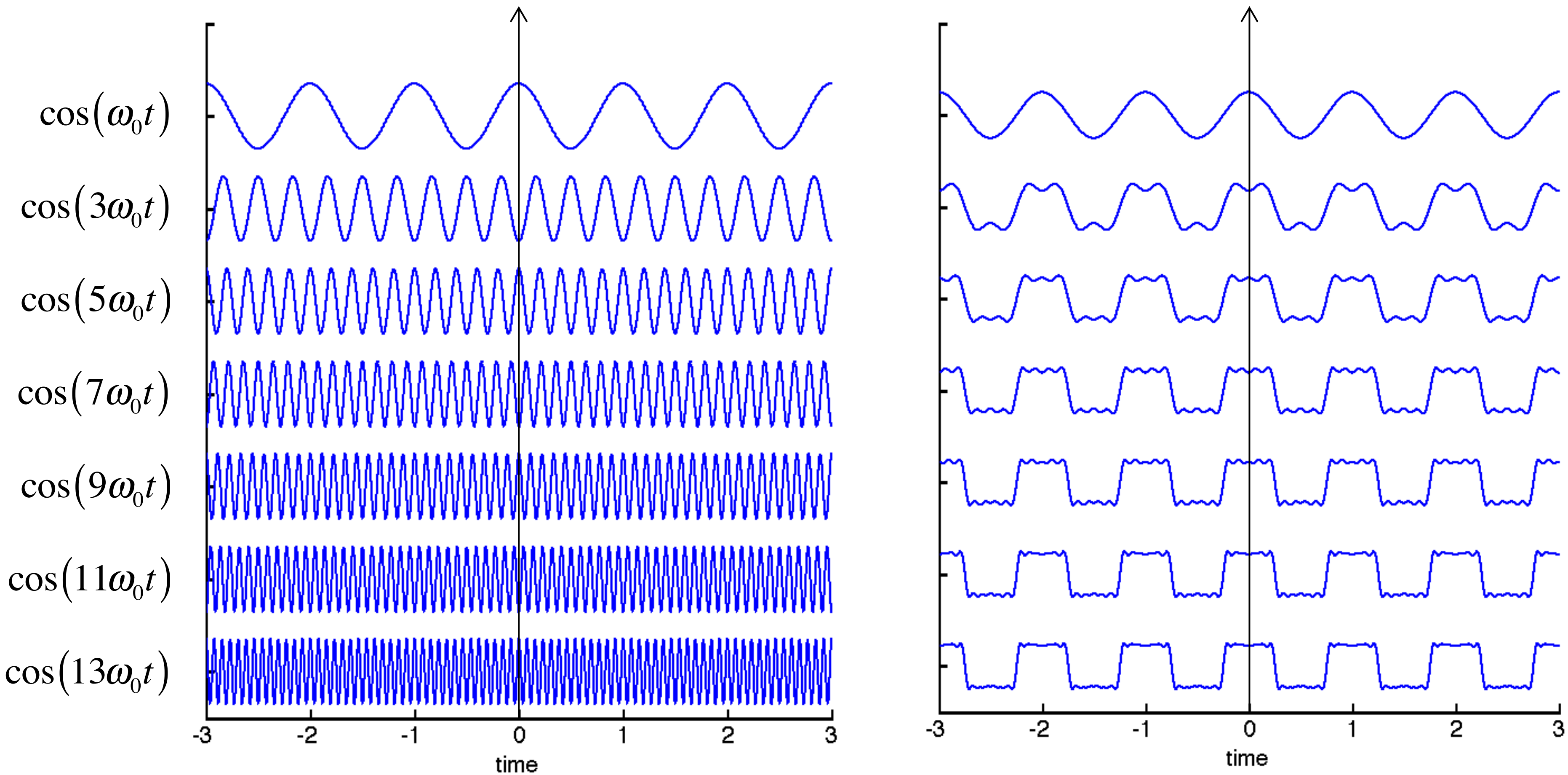
# Fourier Series

- But we can get a better approximation if we add some more cosine waves to our original one…




$$y(t) = a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + a_3 \cos(3\omega_0 t) + \ldots$$

# Fourier Series



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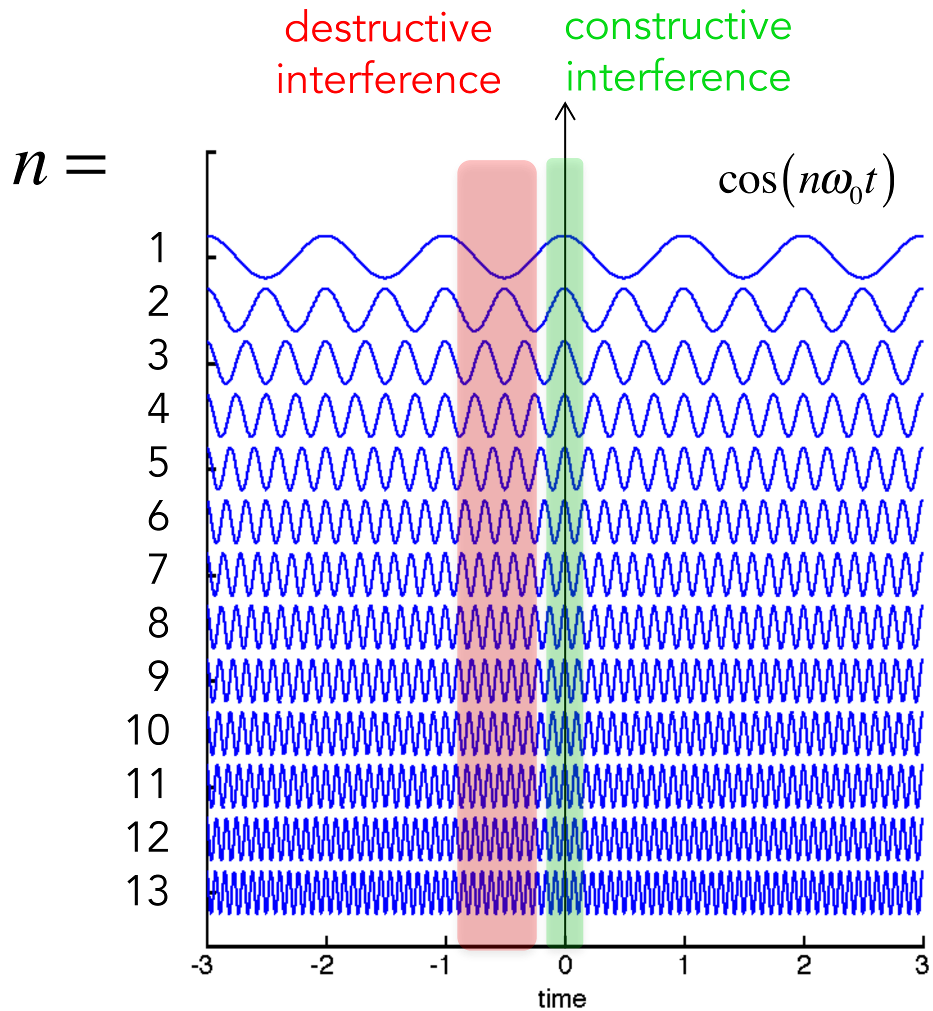
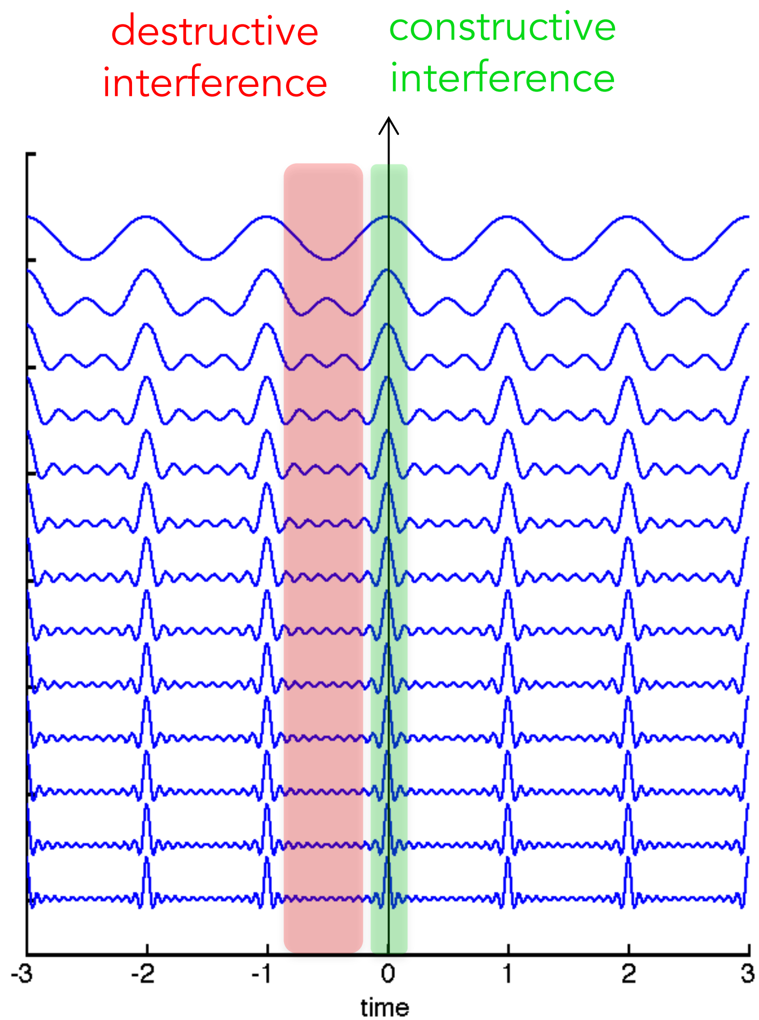
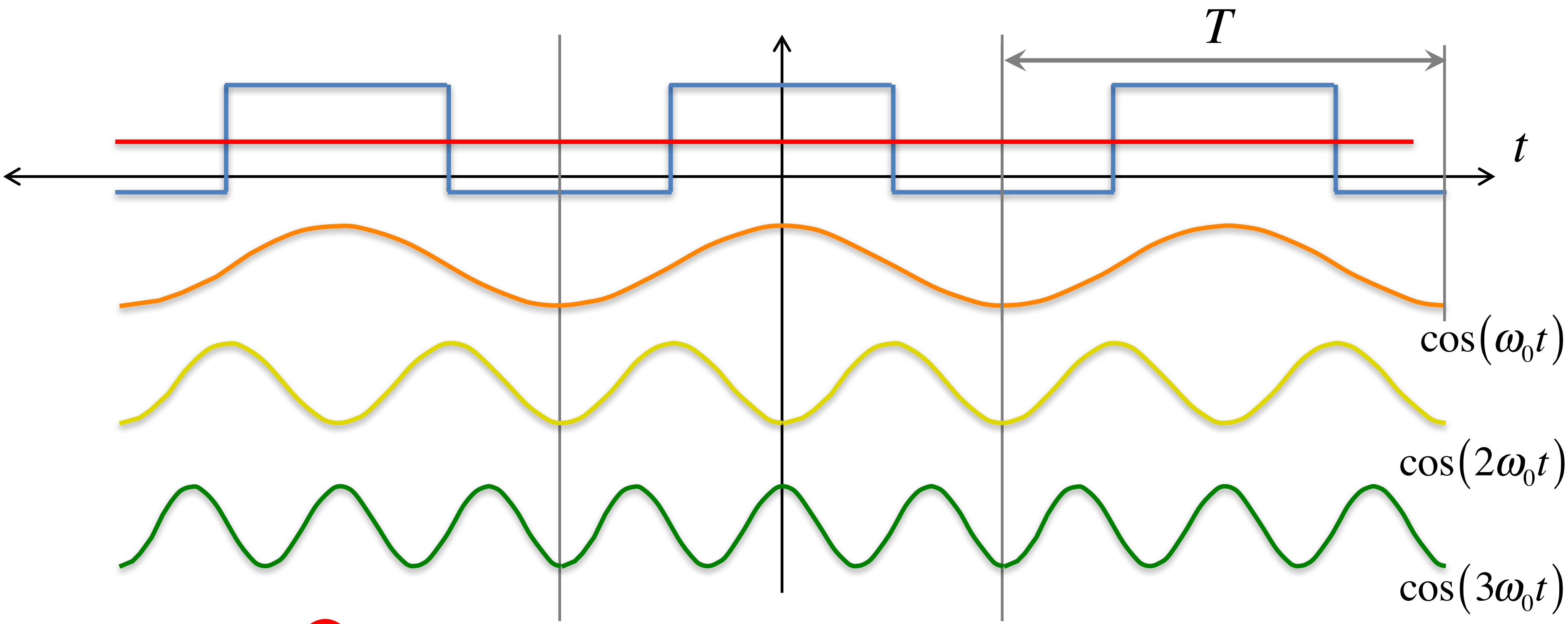


# Fourier Series




$$y(t) = \frac{a_0}{2} + a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + a_3 \cos(3\omega_0 t) + \ldots$$

DC term

$$y_{even}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t)$$

# How do we find the coefficients?

- The $a_0$ coefficient is just like the average of our function y(t).

$$\frac{a_0}{2} = \frac{1}{T}\int_{-T/2}^{T/2} y(t)\,dt \qquad a_0 = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(0\omega_0 t)\,dt$$

- The $a_1$ coefficient is just the **overlap** of our function y(t) with $\cos(\omega_0 t)$

$$a_1 = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(\omega_0 t)\,dt$$

← Correlation!

- The $a_2$ coefficient is just the overlap of our function y(t) with $\cos(2\omega_0 t)$

$$a_2 = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(2\omega_0 t)\,dt$$

- The $a_n$ coefficient is just the overlap of our function y(t) with $\cos(n\omega_0 t)$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(n\omega_0 t)\,dt$$

# How do we find the coefficients?

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\,dt \qquad a_1 = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(\omega_0 t)\,dt \qquad a_2 = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(2\omega_0 t)\,dt$$

Consider the following functions y(t):

| | | | |
|---|---|---|---|
| $y(t)=1$ | $a_0=2$ | $a_1=0$ | $a_2=0$ |
| $y(t)=\cos(\omega_0 t)$ | $a_0=0$ | $a_1=1$ | $a_2=0$ |
| $y(t)=\cos(2\omega_0 t)$ | $a_0=0$ | $a_1=0$ | $a_2=1$ |

$$\int_{-T/2}^{T/2} \left[\cos(\omega_0 t)\right]^2 dt = \frac{T}{2} \qquad \int_{-T/2}^{T/2} \cos(\omega_0 t)\cos(2\omega_0 t)\,dt = 0$$

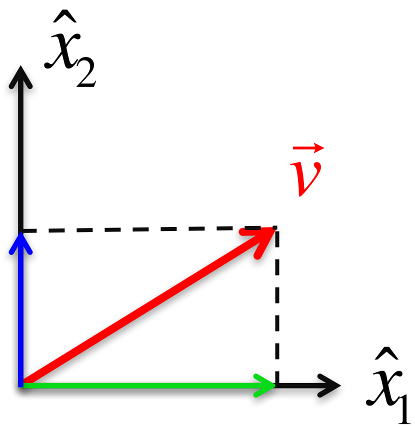
$$y(t) = \frac{a_0}{2} + a_1\cos(\omega_0 t) + a_2\cos(2\omega_0 t) + \ \ldots$$

# Fourier Series

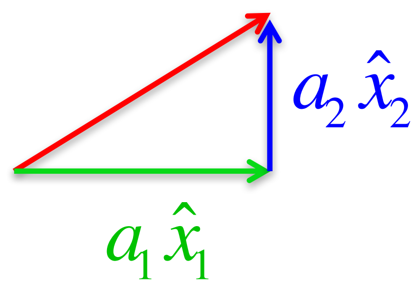
- If a function has maximal overlap with one of our cosine functions, then it has zero overlap with all the others!

- We say that our set of cosine functions form an orthogonal basis set…

$$u_n(t) = \cos(n\omega_0 t)$$




$$\vec{v} = a_1\,\hat{x}_1 + a_2\,\hat{x}_2$$




$$\hat{x}_1 = [0\ ,1]$$

$$\hat{x}_2 = [1\ ,0]$$

$$\vec{v} = [a_1\ ,a_2]$$

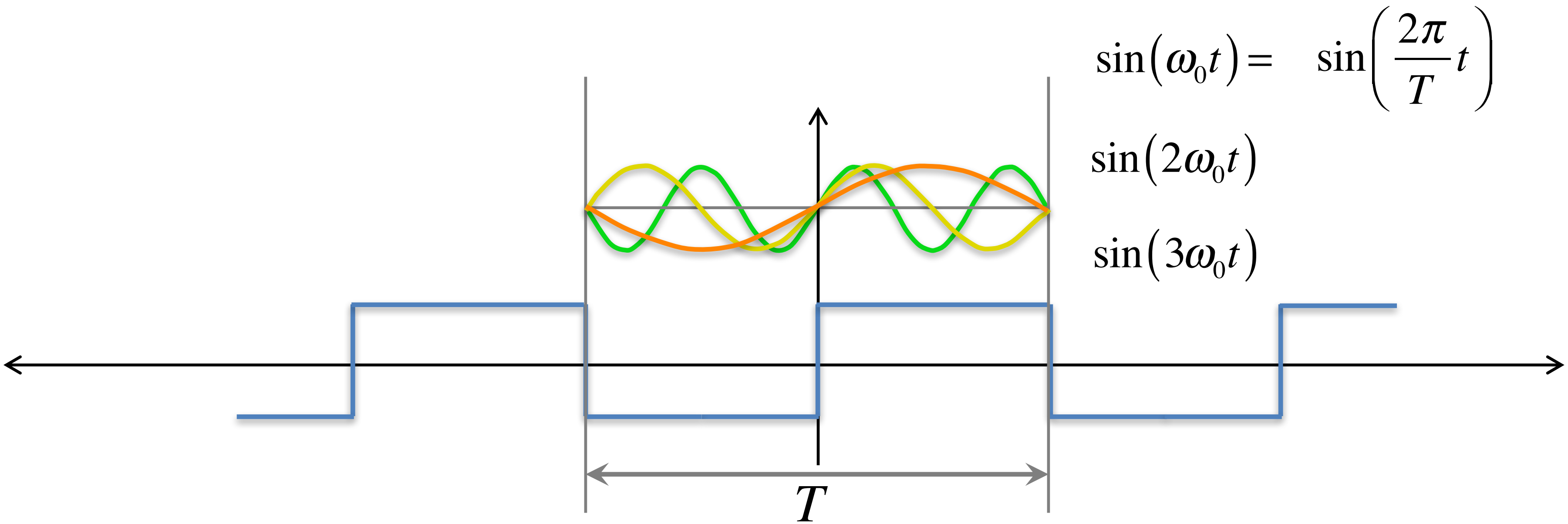
How do we find the coefficients $a_1$ and $a_2$?

$$a_1 = \vec{v}\cdot\hat{x}_1 = \sum_i v^i x_1^i \qquad a_2 = \vec{v}\cdot\hat{x}_2 = \sum_i v^i x_2^i$$

$$a_1 = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos\left(\omega_0 t\right)dt$$

# Fourier Series

- Now let's look an an odd (antisymmetric) function…




$$y_{odd}(t) = b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + b_3 \sin(3\omega_0 t) + \ldots$$

$$y_{odd}(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

Why is there no DC term here?

# Fourier Series

- For an arbitrary function, we can write it down as the sum of a symmetric and an antisymmetric part.

$$y(t) = \underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\omega_0 t\right)}_{\text{symmetric}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin\left(n\omega_0 t\right)}_{\text{antisymmetric}}$$

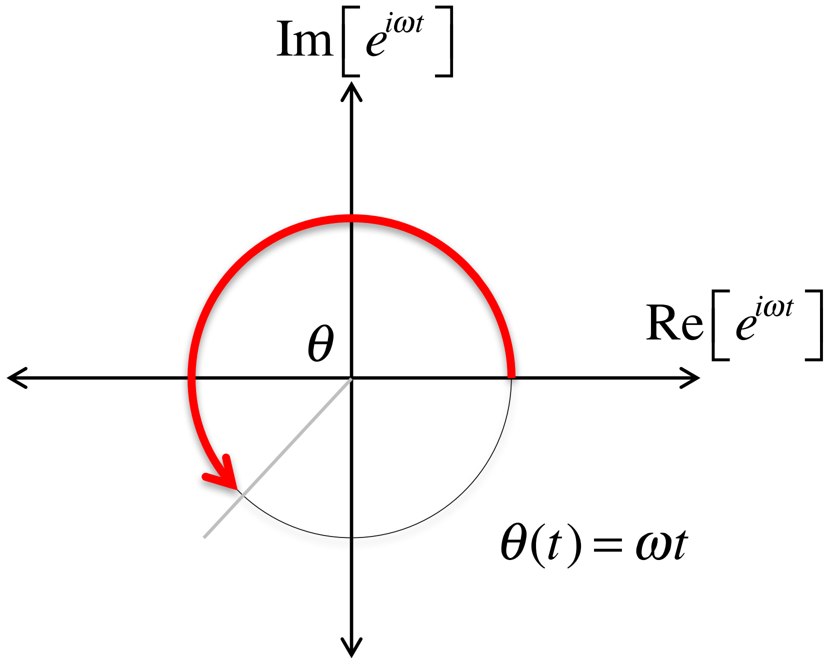
# Complex Fourier Series

- We can express any periodic function of time as sums of complex exponentials.

Euler's formula

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

$$e^{-i\omega t} = \cos\omega t - i\sin\omega t$$




Rewrite as follows…

$$\cos\omega t = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$$\sin\omega t = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right) = -\frac{i}{2}\left(e^{i\omega t} - e^{-i\omega t}\right)$$

$$\frac{1}{i} = -i$$

# Fourier Series

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\omega_0 t\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\omega_0 t\right)$$

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{in\omega t} + e^{-in\omega t}\right) + \sum_{n=1}^{\infty} \frac{-ib_n}{2}\left(e^{in\omega t} - e^{-in\omega t}\right)$$

$$y(t) = A_0 + \sum_{n=1}^{\infty} A_n e^{in\omega_0 t} + \sum_{n=1}^{\infty} A_{-n} e^{-in\omega_0 t}$$

'DC' or 'constant' term

positive frequencies

negative frequencies

$$A_0 = \frac{a_0}{2} \qquad A_n = \frac{1}{2}\left(a_n - ib_n\right) \qquad A_{-n} = \frac{1}{2}\left(a_n + ib_n\right) \qquad A_n = \left(A_{-n}\right)^*$$

complex conjugates

# Complex Fourier Series

$$y(t) = A_0 + \sum_{n=1}^{\infty} A_n e^{in\omega_0 t} + \sum_{n=1}^{\infty} A_{-n} e^{-in\omega_0 t}$$

- We can write this more compactly as follows:

$$= \sum_{n=0} A_n e^{in\omega_0 t} + \sum_{n=1}^{\infty} A_n e^{in\omega_0 t} + \sum_{n=-1}^{-\infty} A_n e^{in\omega_0 t}$$

$$For \;\; n = 0,$$

$$e^{in\omega_0 t} = e^0 = 1$$

$$y(t) = \sum_{n=-\infty}^{\infty} A_n e^{in\omega_0 t}$$

# Learning objectives for Lecture 10

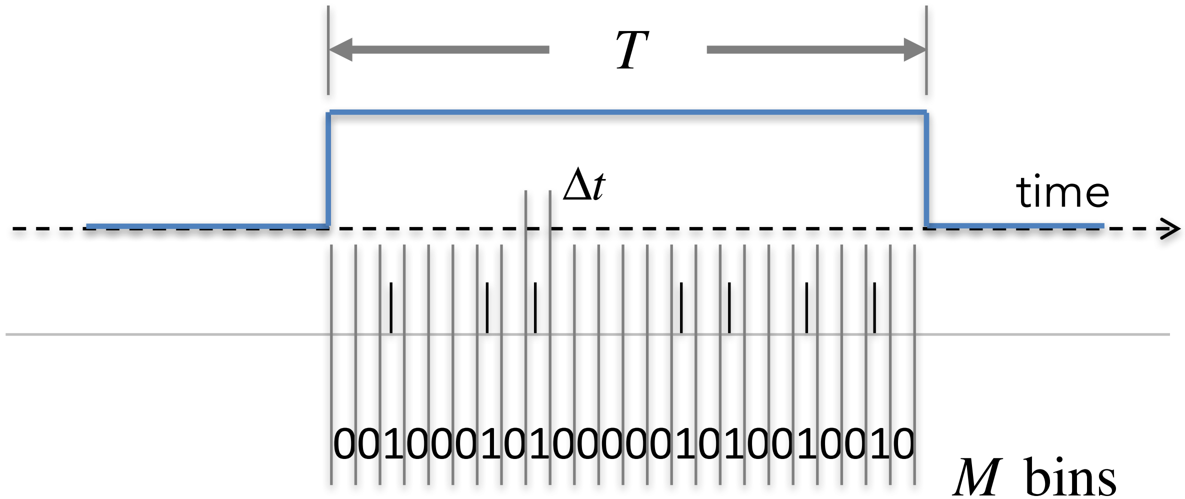
- Spike trains are probabilistic (Poisson Process)

- Be able to use measures of spike train variability
  - Fano Factor
  - Interspike Interval (ISI)

- Understand convolution, cross-correlation, and autocorrelation functions

- Understand the concept of a Fourier series

# Extra Slides on Poisson process




How many spikes land in the interval T ?

What is the probability that n spikes land in the interval T ? $P_T[n]$

This is just the product of three things:

- The probability of having n bins with a spike $= (\mu\,\Delta t)^n$
- The probability of having M-n bins with no spike $= (1-\mu\,\Delta t)^{M-n}$
- The number of different ways to distribution n spikes in M bins $= \dfrac{M!}{(M-n)!n!}$

# Extra Slides on Poisson process

What is the probability that n spikes land in the interval T ?

$$P_T[n] = \lim_{\Delta t \to 0} \frac{M!}{(M-n)!n!} (\mu \Delta t)^n (1-\mu \Delta t)^{M-n}$$

Note that as $\Delta t \to 0$: $\quad M = \frac{T}{\Delta t} \to \infty \qquad M - n \simeq M$

$$\varepsilon = -\mu \Delta t \qquad \frac{1}{\Delta t} = \frac{-\mu}{\varepsilon}$$

$$\lim_{\Delta t \to 0} (1-\mu \Delta t)^{M-n} = \lim_{\Delta t \to 0} (1-\mu \Delta t)^{\frac{T}{\Delta t}} = \lim_{\varepsilon \to 0} (1+\varepsilon)^{\frac{-\mu T}{\varepsilon}} = \lim_{\varepsilon \to 0} \left[ (1+\varepsilon)^{\frac{1}{\varepsilon}} \right]^{-\mu T}$$

$$= e^{-\mu T}$$

# Extra Slides on Poisson process

What is the probability that n spikes land in the interval T ?

$$P_T[n] = \lim_{\Delta t \to 0} \frac{M!}{(M-n)!\,n!} (\mu \Delta t)^n e^{-\mu T}$$

Note that as $M \to \infty$:

n terms

$$\frac{M!}{(M-n)!} = \underbrace{M(M-1)(M-2)\cdots(M-n+1)} \approx M^n = \left(\frac{T}{\Delta t}\right)^n$$

$$P_T[n] = \frac{1}{n!} \left(\frac{T}{\Delta t}\right)^n (\mu \Delta t)^n e^{-\mu T}$$

Poisson distribution!

$$P_T[n] = \frac{(\mu T)^n}{n!} e^{-\mu T}$$