Brief Notes #10

Point and Interval Estimation of Distribution Parameters

(a) Some Common Distributions in Statistics

- **Chi-square distribution**

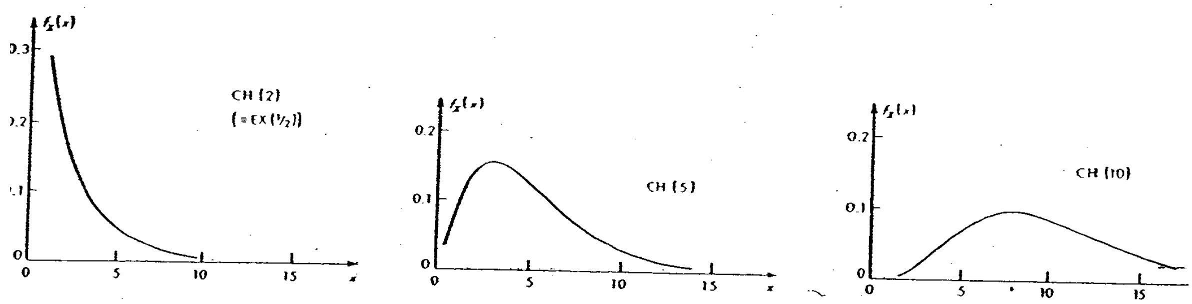
Let $Z_1, Z_2, \ldots, Z_n$ be iid standard normal variables. The distribution of

$$\chi_n^2 = \sum_{i=1}^{n} Z_i^2$$

is called the Chi-square distribution with *n* degrees of freedom.

$$E[\chi_n^2] = n$$

$$Var[\chi_n^2] = 2\,n$$



Probability density function of χ_n^2 for n = 2, 5, 10.

- **t distribution**

Let $Z, Z_1, Z_2, \ldots, Z_n$ be iid standard normal variables. The distribution of

$$t_n = \frac{Z}{\left(\frac{1}{n}\sum_{i=1}^{n} Z_i^2\right)^{1/2}}$$

is called the Student's t distribution with *n* degrees of freedom.

$E[t_n] = 0$

$$\mathrm{Var}[t_n] = \frac{n}{n-2}, \qquad n > 2$$
$$= \infty, \qquad n \le 2$$

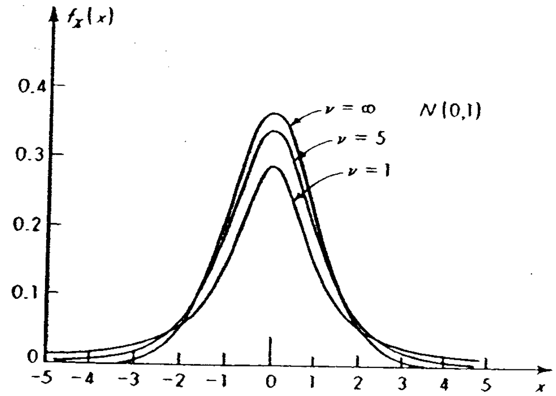


Fig. 3.4.4 The *t* distribution. (From A. Hald, "Statistical Theory with Engineering Applications," John Wiley and Sons, New York, 1952.)

Probability density function of t_n for $n = 1, 5, \infty$.
Note: $t_\infty = N(0, 1)$.

- **F distribution**

Let $W_1, W_2, \ldots, W_n, Z_1, Z_2, \ldots, Z_n$ be iid standard normal variables. The distribution of

$$F_{m,n} = \frac{\frac{1}{m}\sum_{i=1}^{n} W_i^2}{\frac{1}{n}\sum_{i=1}^{n} Z_i^2} = \frac{\frac{1}{m}\chi_m^2}{\frac{1}{n}\chi_n^2}$$

is called the F distribution with *m* and *n* degrees of freedom.

As $n \to \infty$, $m\,F_{m,n} \to \chi_m^2$

(b) Point Estimation of Distribution Parameters: Objective and Criteria

- **Definition of (point) estimator**

  Let θ be an unknown *parameter* of the distribution F_X of a random variable X, for example the mean m or the variance σ^2. Consider a random *sample* of size n from the statistical *population* of X, $\{X_1, X_2, \ldots, X_n\}$. An estimator $\hat{\Theta}$ of θ is a function $\hat{\Theta}(X_1, X_2, \ldots, X_n)$ that produces a numerical estimate of θ for each realization $x_1, x_2, \ldots, x_n$ of $X_1, X_2, \ldots, X_n$. Notice: $\hat{\Theta}$ is a random variable whose distribution depends on θ.

- **Desirable properties of estimators**

  1. *Unbiasedness*:
     $\hat{\Theta}$ is said to be an unbiased estimator of θ if, for any given θ, $E_{sample}[\hat{\Theta} \mid \theta] = \theta$.
     The bias $b_{\hat{\Theta}}(\theta)$ of $\hat{\Theta}$ is defined as:

     $$b_{\hat{\Theta}}(\theta) = E_{sample}[\hat{\Theta} \mid \theta] - \theta$$

  2. *Mean Squared Error (MSE)*:
     The mean squared error of $\hat{\Theta}$ is the second initial moment of the estimation error $e = \hat{\Theta} - \theta$, i.e.,

     $$MSE_{\hat{\Theta}}(\theta) = E[(\hat{\Theta} - \theta)^2] = b_{\hat{\Theta}}^2(\theta) + Var[\hat{\Theta} \mid \theta]$$

     One would like the mean square error of an estimator to be as small as possible.

(c) Point Estimation of Distribution Parameters: Methods

1. Method of moments

Suppose that F_X has unknown parameters $\theta_1, \theta_2, \ldots, \theta_r$. The idea behind the method of moments is to estimate $\theta_1, \theta_2, \ldots, \theta_r$ so that r selected characteristics of the distribution match their sample values. The characteristics are often taken to be the initial moments:

$$\mu_i = E[X^i], \qquad i = 1, \ldots, r$$

The method is described below for the case $r = 2$.

The first and second initial moments of X are, in general, functions of the unknown parameters, θ_1 and θ_2:

$$\mu_1(\theta_1,\theta_2) = E[X \mid \theta_1,\theta_2] = \int x \, f_{X|\theta_1,\theta_2}(x)\, dx$$

$$\mu_2(\theta_1,\theta_2) = E[X^2 \mid \theta_1,\theta_2] = \int x^2 \, f_{X|\theta_1,\theta_2}(x)\, dx$$

The sample values of these moments are:

$$\hat{\mu}_1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}$$

$$\hat{\mu}_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

Estimators of θ_1 and θ_2 are obtained by solving the equations for $\hat{\Theta}_1$ and $\hat{\Theta}_2$:

$$\mu_1(\hat{\Theta}_1,\hat{\Theta}_2) = \hat{\mu}_1$$

$$\mu_2(\hat{\Theta}_1,\hat{\Theta}_2) = \hat{\mu}_2$$

This method is often simple to apply, but may produce estimators that have higher MSE than other methods, e.g. maximum likelihood.

Example:
If $\theta_1 = m$ and $\theta_2 = \sigma^2$, then:

$\mu_1 = m$ and $\mu_2 = m^2 + \sigma^2$

$\hat{\mu}_1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}$ and $\hat{\mu}_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$

The estimators $\hat{m}$ and $\hat{\sigma}^2$ are obtained by solving:

$$\hat{m} = \overline{X}$$

$$\hat{m}^2 + \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

which gives:

$$\hat{m} = \overline{X}$$

$$
\begin{aligned}
\hat{\sigma}^2 &= \left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) - \overline{X}^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2
\end{aligned}
$$

Notice that $\hat{\sigma}^2$ is a biased estimator since its expected value is $\frac{n-1}{n}\sigma^2$. For this reason, one typically uses the modified estimator:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

which is unbiased.

2. **Method of maximum likelihood:**

Consider again the case $r = 2$. The likelihood function of θ_1 and θ_2 given a sample, $L(\theta_1, \theta_2 \mid \text{sample})$, is defined as:

$$L(\theta_1, \theta_2 \mid \text{sample}) \propto P[\text{sample} \mid \theta_1, \theta_2]$$

Where P is either probability or probability density and is regarded for a given sample as a function of θ_1 and θ_2. In the case when X is a continuous variable:

$$P[\text{sample} \mid \theta_1, \theta_2] = \prod_{i=1}^{n} f_X(x_i \mid \theta_1, \theta_2)$$

The maximum likelihood estimators $(\hat{\Theta}_1)_{ML}$ and $(\hat{\Theta}_2)_{ML}$ are the values of θ_1 and θ_2 that maximize the likelihood, i.e.,

$L(\theta_1, \theta_2 \mid \text{sample})$ is maximum for $\theta_1 = (\hat{\Theta}_1)_{ML}$ and $\theta_2 = (\hat{\Theta}_2)_{ML}$

In many cases, $(\hat{\Theta}_1)_{ML}$ and $(\hat{\Theta}_2)_{ML}$ can be found by imposing the stationarity conditions:

$$\frac{\partial L[(\hat{\Theta}_1, \hat{\Theta}_2) \mid \text{sample}]}{\partial \hat{\Theta}_1} = 0 \qquad \text{and} \qquad \frac{\partial L[(\hat{\Theta}_1, \hat{\Theta}_2) \mid \text{sample}]}{\partial \hat{\Theta}_2} = 0$$

or, more frequently, the equivalent conditions in terms of the log-likelihood:

$$\frac{\partial\{\ln L[(\hat{\Theta}_1,\hat{\Theta}_2) \mid \text{sample}]\}}{\partial\hat{\Theta}_1} = 0 \text{ and } \frac{\partial\{\ln L[(\hat{\Theta}_1,\hat{\Theta}_2) \mid \text{sample}]\}}{\partial\hat{\Theta}_2} = 0$$

<u>Properties of maximum likelihood estimators:</u>

As the sample size $n \to \infty$, maximum likelihood estimators:

1. are unbiased;
2. have the smallest possible value of MSE.

<u>Example:</u>

For $X \sim N(m, \sigma^2)$ with unknown parameters m and σ^2, the maximum likelihood estimators of the parameters are:

$$\hat{m}_{ML} = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X} \quad \sim N(m, \frac{\sigma^2}{n})$$

$$\hat{\sigma}^2_{ML} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{m}_{ML})^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2 \quad \sim \frac{\sigma^2}{n}\chi^2_{(n-1)}$$

Notice that in this case the ML estimators m and σ^2 are the same as the estimators produced by the method of moments. This is not true in general.

3. Bayesian estimation

The previous two methods of point estimation are based on the classical statistical approach which assumes that the distribution parameters $\theta_1, \theta_2, \ldots, \theta_r$ are constants but unknown. In Bayesian estimation, $\theta_1, \theta_2, \ldots, \theta_r$ are viewed as uncertain (random variables) and their uncertainty is quantified through probability distributions. There are 3 steps in Bayesian estimation:

<u>Step 1:</u> Quantify initial uncertainty on $\underline{\theta}$ in the form of a prior distribution, $f'_{\underline{\theta}}$

<u>Step 2:</u> Use sample information to update uncertainty $\to$ posterior distribution, $f''_{\underline{\theta}}$

<u>Step 3:</u> Choose a single value estimate of $\underline{\theta}$

$$\begin{tikzcd} f'_{\underline{\theta}} \arrow[r] & f''_{\underline{\theta}} = f_{\underline{\theta}|X_1,\ldots,X_n} \arrow[r] & \hat{\underline{\theta}} \\ \text{Sample } \{X_1,\ldots,X_n\} \arrow[ur] & & \text{Consequences of estimation errors } \hat{\underline{\theta}} - \underline{\theta} \arrow[u] \end{tikzcd}$$

The various steps are described below in the order 2, 3, 1.

Step 2: **How to update prior uncertainty given a sample**

Recall that for random variables,

$$f_{\underline{\theta}|\underline{X}} \propto f_{\underline{\theta}}(\underline{\theta}) \cdot f_{\underline{X}|\underline{\theta}}(\underline{x})$$

Here, $f'_{\underline{\theta}} = f_{\underline{\theta}}$ and $f''_{\underline{\theta}} = f_{\underline{\theta}|\underline{X}}$. Further, using $\ell(\underline{\theta} \mid \underline{X}) \propto f_{\underline{X}|\underline{\theta}}(\underline{x})$, one obtains:

$$f''_{\underline{\theta}}(\underline{\theta}) \propto f'_{\underline{\theta}}(\underline{\theta}) \ell(\underline{\theta} \mid \underline{X})$$

Step 3: **How to choose $\hat{\underline{\theta}}$**

Two main methods:

1. *Use some characteristic of* $f''_{\underline{\theta}}$, such as the mean or the mode. The choice is rather arbitrary. Note that the mode corresponds in a sense to the maximum likelihood, applied to the posterior distribution rather than the likelihood.
2. *Decision theoretic approach: (more objective and preferable)*
   θ by $\hat{\theta}$.

- Define a loss function $\$(\hat{\theta} \mid \theta)$ which is the loss if the estimate is $\hat{\underline{\theta}}$ and the true value is $\underline{\theta}$.
- Calculate the expected posterior loss or "Risk" of $\hat{\underline{\theta}}$ as:

$$R(\hat{\underline{\theta}}) = E''[\$(\hat{\underline{\theta}} \mid \underline{\theta})] = \int_{-\infty}^{\infty} \$(\hat{\underline{\theta}} \mid \underline{\theta}) f''_{\underline{\theta}}(\underline{\theta}) d\underline{\theta}$$

– Choose $\hat{\underline{\theta}}$ such that $R(\hat{\underline{\theta}})$ is **minimum**.

- If $\$(\hat{\underline{\theta}}, \underline{\theta})$ is a quadratic function of $(\hat{\theta}_i - \theta_i)$, then $R(\hat{\underline{\theta}})$ is minimum for $\hat{\underline{\theta}} = E''[\underline{\theta}]$
- If $\$(\hat{\underline{\theta}}, \underline{\theta}) = \begin{cases} 0, & \text{if } \hat{\underline{\theta}} = \underline{\theta} \\ c > 0, & \text{if } \hat{\underline{\theta}} \neq \underline{\theta} \end{cases}$, then $\hat{\underline{\theta}}$ is the mode of $f''_{\underline{\theta}}$.

Step 1: **How to select $f'_{\underline{\theta}}$**

1. *Judgmentally.* This approach is especially useful in engineering design, where subjective judgment is often necessary. This is how subjective judgment is formally incorporated in the decision process.

2. Based on *prior data* e.g. a "sample" of $\underline{\theta}$'s from other data sets
3. To *reflect ignorance*, "non-informative prior".
   For example, if θ is a scalar parameter that can attain values from $-\infty$ to $+\infty$, then $f'_\theta(\theta)d\theta \propto d\theta$ ("flat") and $f''_\theta(\theta) \propto \ell(\theta \mid \text{sample})$ i.e. the posterior reflects only the likelihood.

   If $\theta > 0$, then one typically takes $f'_{\ln\theta}(\ln\theta)d\ln\theta \propto d\ln\theta$. In this case, $f'_\theta(\theta) \propto \frac{1}{\theta}$.
4. *Conjugate prior*. There are distribution types such that if $f'_\theta(\theta)$ is of that type, then $f''_\theta(\theta) \propto f'_\theta(\theta)\ell(\theta)$ is also of the same type. Such distributions are called conjugate distributions.

Example:

Let:
$X \sim N(m,\sigma^2)$ with σ^2 known. $\theta = m$ unknown.
Suppose: $f'_m \sim N(m',\sigma'^2)$
It can be shown that $\ell(m \mid X_1,\ldots,X_n) \propto$ density of $N(\overline{X},\sigma^2/n)$
From $f''_m \propto f'_m \ell(m \mid \text{sample})$, one obtains

$$f''_m \sim N\left(m'' = \frac{m'(\sigma^2/n) + \overline{X}\sigma'^2}{(\sigma^2/n) + \sigma'^2}, \frac{1}{\sigma''^2} = \frac{1}{\sigma'^2} + \frac{n}{\sigma^2}\right)$$

In this case, $f'_m \sim N(m',\sigma'^2)$ is an example of a conjugate prior, since f''_m is also normal, of the type $N(m'',\sigma''^2)$.

If one writes $\sigma'^2 = \frac{\sigma^2}{n'}$, then n' has the meaning of equivalent prior sample size and m' has the meaning of equivalent prior sample average.

(d) Approximate Confidence Intervals for Distribution Parameters

1. Classical Approach

Problem: θ is an unknown distribution parameter. Define two sample statistics $\hat{\Theta}_1(X_1, \ldots, X_n)$ and $\hat{\Theta}_2(X_1, \ldots, X_n)$ such that:

$$P[\hat{\Theta}_1(X_1, \ldots, X_n) < \theta < \hat{\Theta}_2(X_1, \ldots, X_n)] = P^*$$

where P* is a given probability.

An interval $[\hat{\Theta}_1(X_1, \ldots, X_n), \hat{\Theta}_2(X_1, \ldots, X_n)]$ with the above property is called a <u>confidence interval</u> of θ at confidence level P*.

A simple method to obtain confidence intervals is as follows. Consider a point estimation $\hat{\Theta}$ such that, exactly or in approximation, $\hat{\Theta} \sim N(\theta, \sigma^2(\theta))$. If the variance $\sigma^2(\theta)$ depends on θ, one replaces $\sigma^2(\theta)$ with $\sigma^2(\hat{\Theta})$. Then:

$$\frac{\hat{\Theta}-\theta}{\sigma(\hat{\Theta})} \sim N(0, 1)$$

$$\Rightarrow \qquad P[\hat{\Theta} - \sigma(\hat{\Theta})\, Z_{P^*/2} < \theta < \hat{\Theta} + \sigma(\hat{\Theta})\, Z_{P^*/2}] = P^*$$

where Z_α is the value exceeded with probability α by a standard normal variable.

<u>Example:</u>
$\theta = m =$ mean of an exponential distribution.

In this case, $\hat{\Theta} = \overline{X} \sim \frac{1}{n}\text{Gamma}(m, n)$, where Gamma (m, n) is the distribution of the sum of n iid exponential variables, each with mean value m. The mean and variance of Gamma(m, n) are nm and nm^2, respectively. Moreover, for large n, Gamma(m, n) is close to $N(nm, nm^2)$. Therefore, in approximation,

$$\overline{X} \sim N(m, \frac{m^2}{n})$$

Using the previous method, an approximate confidence interval for m at confidence level P* is

$$[\overline{X} - \frac{\overline{X}}{\sqrt{n}} \cdot Z_{P^*/2}, \overline{X} + \frac{\overline{X}}{\sqrt{n}} \cdot Z_{P^*/2}]$$

2. *Bayesian Approach*

In Bayesian analysis, intervals $[\hat{\theta}_1, \hat{\theta}_2]$ that contain θ with a given probability P* are simply obtained from the condition that:

$$F_\theta''(\hat{\theta}_2) - F_\theta''(\hat{\theta}_1) = P^*$$

where F_θ'' is the posterior CDF of θ.