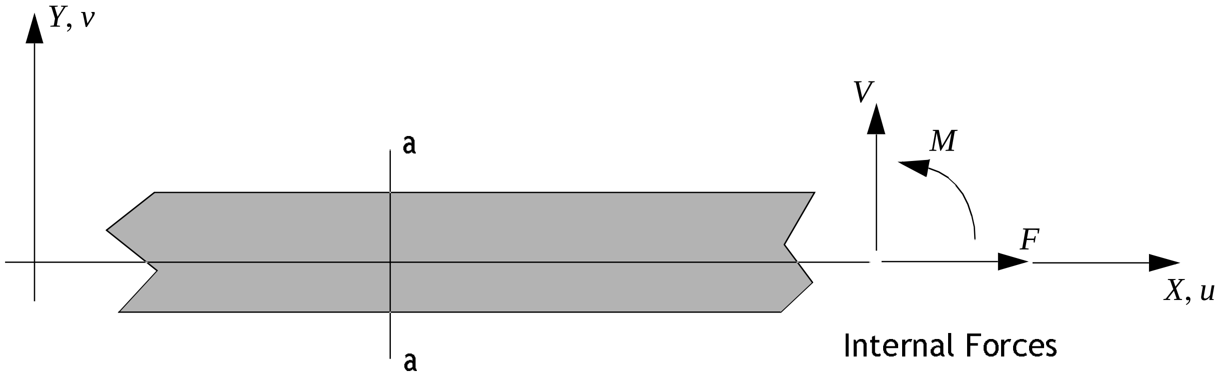
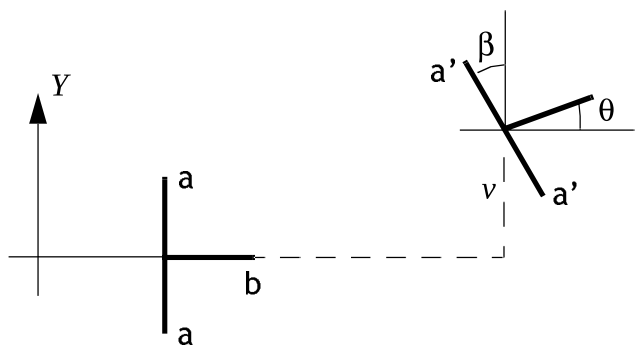
1.571 Structural Analysis and Control
Prof. Connor
Section 1: Straight Members with Planar Loading

Governing Equations for Linear Behavior

1.1 Notation



1.1.2 Deformation - Displacement Relations



Assume β is small

Longitudinal strain at location y:

$$\varepsilon(y) = \frac{\partial}{\partial x}u(y)$$

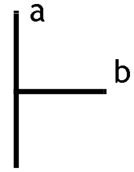
For small β

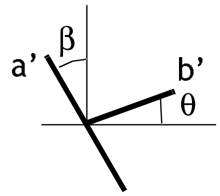
$$u(y) \approx u(0) - y\beta$$
$$v(y) \approx v(0)$$

Then

$$\varepsilon(y) = u_{,x} - y\beta_{,x} = \varepsilon_a + \varepsilon_b$$
$$\varepsilon_a = u_{,x} = \text{stretching strain}$$
$$\varepsilon_b = -y\beta_{,x} = \text{bending strain}$$

Shear Strain





$$\gamma = \text{decrease in angle between lines a and b}$$

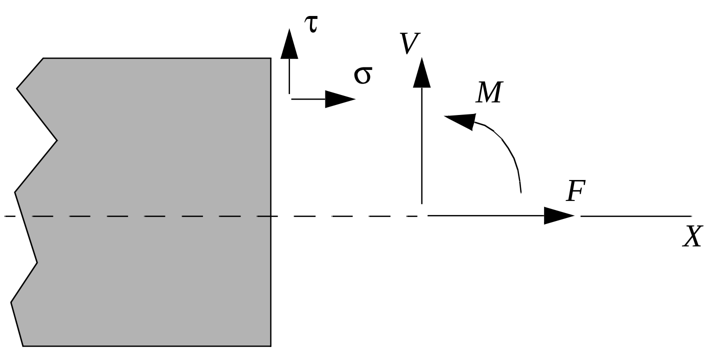
$$\gamma = \theta - \beta$$

$$\theta \approx \frac{dv}{dx} = v,_x$$

$$\gamma = v,_x - \beta$$

1.3 Force - Deformation Relations

$$\left.\begin{array}{l}\sigma = E\varepsilon \\ \tau = G\gamma\end{array}\right\}$$ stress strain relations for linear elastic material



$$F = \int \sigma dA$$

$$M = \int -y\sigma dA$$

$$V = \int \tau dA$$

Consider initial strain for longitudinal actions

$$\varepsilon_\sigma + \varepsilon_o = \varepsilon_a + \varepsilon_b = \varepsilon_T$$

where

$$\varepsilon_\sigma = \text{strain due to stress}$$

$$\varepsilon_o = \text{initial strain}$$

$$\varepsilon_T = \text{total strain} = \varepsilon_a + \varepsilon_b$$

Then

$$\varepsilon_\sigma = \varepsilon_T - \varepsilon_o = \frac{1}{E}\sigma$$

$$\sigma = E(\varepsilon_T + \varepsilon_o) = E(\varepsilon_a + \varepsilon_b - \varepsilon_o)$$
$$= E(u,_x - y\beta,_x - \varepsilon_o)$$

$$\therefore \qquad F = \int \sigma dA$$

$$F = \int E(u,_x - y\beta,_x - \varepsilon_o) dA$$

$$F = u,_x \int E dA + \beta,_x \int -yE dA + \int -\varepsilon_o E dA$$

Also

$$M = \int -y\sigma dA$$

$$M = \int -yE(u,_x - y\beta,_x - \varepsilon_o) dA$$

$$M = u,_x \int -yE dA + \beta,_x \int y^2 E dA + \int y\varepsilon_o E dA$$

If one locates the X-axis such that

$$\int yE dA = 0$$

the equations uncouple to give:

$$F = u,_x \int E dA + \int -\varepsilon_o E dA$$

$$M = \beta,_x \int y^2 E dA + \int y\varepsilon_o E dA$$

Define

$$D_S = \int E dA = \text{stretching rigidity}$$

$$D_B = \int y^2 E dA = \text{bending rigidity}$$

$$F_o = -\int \varepsilon_o E dA$$

$$M_o = \int y\varepsilon_o E dA$$

Then

$$\boxed{\begin{aligned} F &= D_S u,_x + F_o \\ M &= D_B \beta,_x + M_o \end{aligned}}$$

Consider no inital shear strain

$$\tau = G\gamma = G(v,_x - \beta)$$

$$V = \int G\gamma dA = \int G(v,_x - \beta) dA$$

$$V = (v,_x - \beta) \int G dA$$

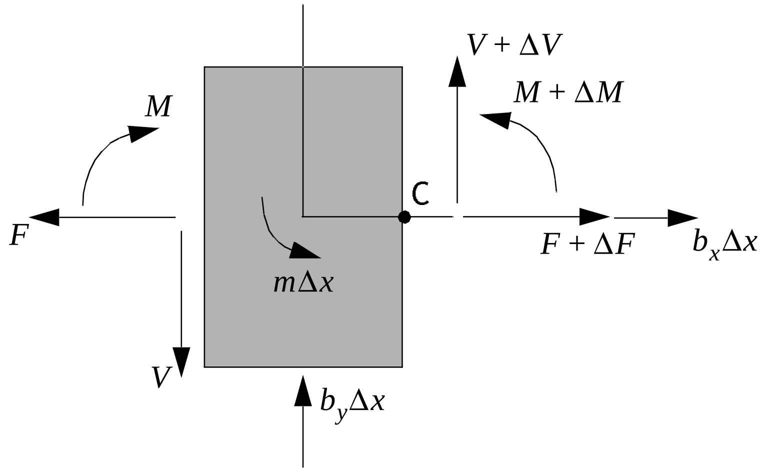
Define

$$D_T = \int G dA = \text{transverse shear rigidity}$$

then

$$V = D_T(v,_x - \beta)$$

1.4 Force Equilibrium Equations



Consider the rate of change of the internal force quantities over an interval Δx

$$\sum F_x = -F + F + \Delta F + b_x\Delta x = 0$$

$$\Delta F + b_x\Delta x = 0$$

$$\frac{\Delta F}{\Delta x} + b_x = 0$$

$$\sum F_y = -V + V + \Delta V + b_y\Delta x = 0$$

$$\Delta V + b_y\Delta x = 0$$

$$\frac{\Delta V}{\Delta x} + b_y = 0$$

$$\sum M_c = -\mathrm{M} + M + \Delta M + m\Delta x - b_y\frac{\Delta x^2}{2} + V\Delta x = 0$$

$$\Delta M + m\Delta x + V\Delta x - b_y\frac{\Delta x^2}{2} = 0$$

$$\frac{\Delta M}{\Delta x} + m + V - b_y\frac{\Delta x}{2} = 0$$

Let $\Delta x \rightarrow 0$ (i.e. $\Delta x \rightarrow dx$)

$$\frac{\partial F}{\partial x} + b_x = 0$$

$$\frac{\partial V}{\partial x} + b_y = 0$$

$$\frac{\partial M}{\partial x} + V + m = 0$$

1.5 Summary of Formulation

Equations "uncouple" into 2 sets of equations; one set for "axial" loading and the other set for "transverse" loading.

Axial (Stretching)

$$F,_x + b_x = 0$$

$$F = F_o + D_S u,_x$$

<u>Boundary Condition</u>

F or u prescribed at each end

Transverse (Bending)

$$V,_x + b_y = 0$$

$$M,_x + V + m = 0$$

$$M = D_B \beta,_x + M_o$$

$$V = D_T(v,_x - \beta)$$

<u>Boundary Conditions</u>

M or β prescribed at each end

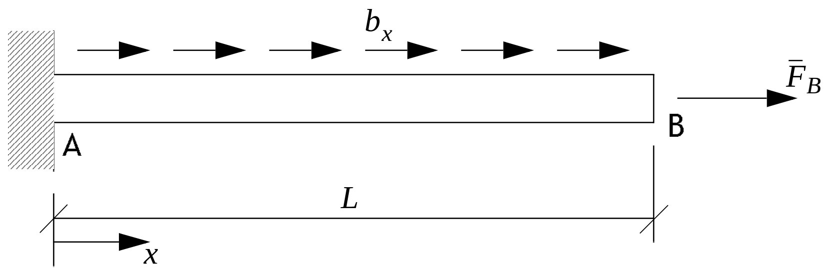
and

V or v prescribed at each end

Note: These equations uncouple for two reasons

1. The location of the X-axis was selected to eliminate the coupling term $\int yEdA$
2. The longitudinal axis is straight and the rotation of the cross-sections is considered to be small. This simplification does not apply when:
   - i - the X-axis is curved (see Section 2)
   - ii - the rotation, β, can not be considered small, creating geometric non-linearity (see Section 4)

1.6 Fundamental Solution - Stretching Problem



Governing Equations:

$$\frac{\partial F}{\partial x} + b_x = 0 \qquad \text{(i)}$$

$$F = F_o + D_S \frac{\partial u}{\partial x} \qquad \text{(ii)}$$

Boundary Conditions

$$F|_B = \bar{F}_B$$

$$u|_A = u_A$$

From (i)

$$F(x) = -\int b_x dx + C_1$$

$$F(x)|_L = -\left(\int b_x dx\right)_L + C_1 = \bar{F}_B$$

$$\therefore \qquad C_1 = \bar{F}_B + \left(\int b_x dx\right)_L$$

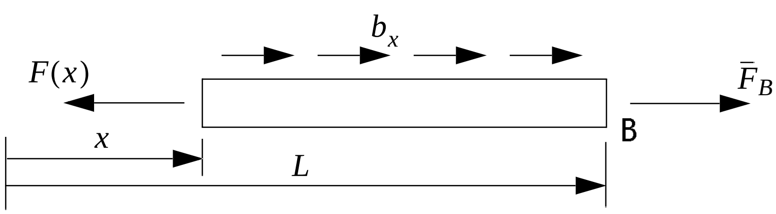
Then

$$F(x) = -\int b_x dx + \left(\int b_x dx\right)_L + \bar{F}_B$$

which can be written as

$$F(x) = \int_x^L b_x dx + \bar{F}_B$$

Note: you could also obtain this result by inspection:



$$F(x) = \int_x^L b_x dx + \bar{F}_B$$

From (ii)

$$\frac{F-F_o}{D_S} = u,_x$$

$$u(x) = \int\frac{F-F_o}{D_S}dx + C_2$$

$$u_A = \left(\int\frac{F-F_o}{D_S}dx\right)_0 + C_2$$

$$C_2 = u_A - \left(\int\frac{F-F_o}{D_S}dx\right)_0$$

$$u(x) = \int_0^x\frac{F-F_o}{D_S}dx + u_A$$

$$u(x) = u_A + \int_0^x\frac{\bar{F}_B}{D_S}dx + u_p(x)$$

$$u(x) = u_A + \frac{\bar{F}_B x}{D_S} + u_p(x)$$

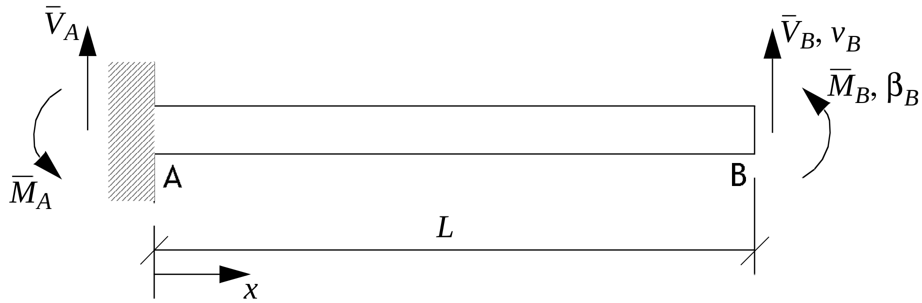
where $u_p(x)$ = particular solution due to b_x and F_o.

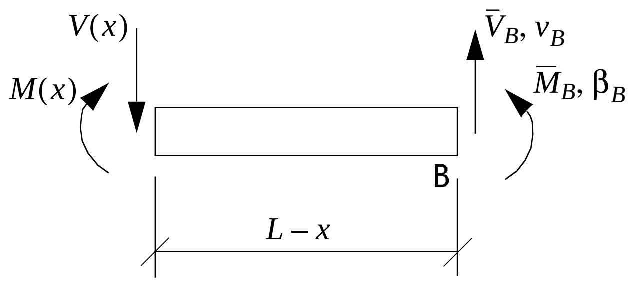
$$\boxed{u_B = u_A + \frac{\bar{F}_B L}{D_S} + u_{B,o}}$$

where

$$u_{B,o} = u_p(L)$$

1.7 Fundamental Solution: Bending Problem





Internal Forces

$$V(x) = \bar{V}_B$$

$$M(x) = \bar{M}_B + \bar{V}_B(L-x)$$

Governing Equations for Displacement

$$M = D_B\beta,_x + M_o \rightarrow \beta,_x = \frac{M - M_o}{D_B}$$

$$V = D_T(v,_x - \beta) \rightarrow v,_x = \beta + \frac{V}{D_T}$$

Integration leads to:

$$\beta(x) = \beta_A + \frac{\bar{M}_B x}{D_B} + \frac{\bar{V}_B}{D_B}\left(Lx - \frac{x^2}{2}\right) + \beta_o(x)$$

$$\boxed{\beta(L) = \beta_B = \beta_A + \frac{\bar{M}_B L}{D_B} + \frac{\bar{V}_B}{D_B}\frac{L^2}{2} + \beta_{B,o}}$$

$$v(x) = v_A + \beta_A L + \frac{\bar{M}_B}{D_B}\frac{x^2}{2} + \frac{\bar{V}_B}{D_B}\left(L\frac{x^2}{2} - \frac{x^3}{6}\right) + \frac{\bar{V}_B}{D_T}x + v_o(x)$$

$$\boxed{v(L) = v_B = v_A + \beta_A L + \frac{\bar{M}_B}{D_B}\frac{L^2}{2} + \frac{\bar{V}_B}{D_B}\frac{L^3}{3} + \frac{\bar{V}_B}{D_T}L + v_{B,o}}$$

1.8 Particular Solutions

Set $\beta_{i,o}$ = end rotation at i due to span load

$v_{i,o}$ = end displacement at i due to span load

Then

$$\beta_i = \beta_{i,e} + \beta_{i,o}$$

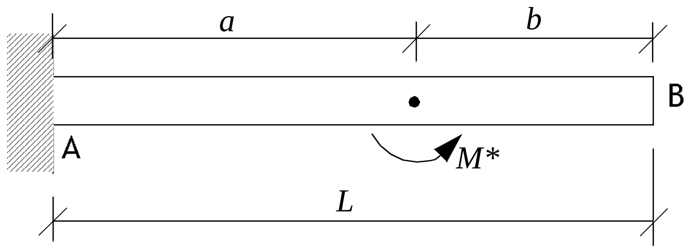
$$v_i = v_{i,e} + v_{i,o}$$

where

$\beta_{i,e}$ = end rotation at i due to end actions

$v_{i,e}$ = end displacement at i due to end actions

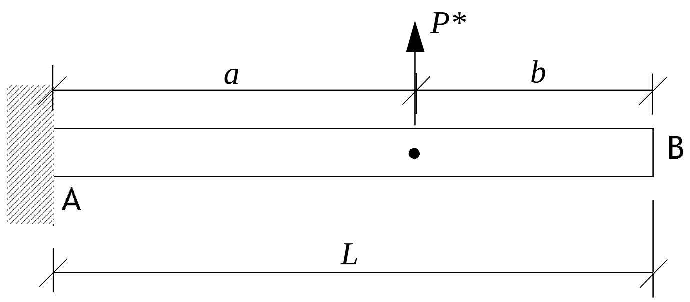
Concentrated Moment



$$\beta_{B,o} = \frac{M^*a}{D_B}$$

$$v_{B,o} = \frac{M^*a^2}{2D_B} + \frac{M^*a}{2D_B}(L-a)$$

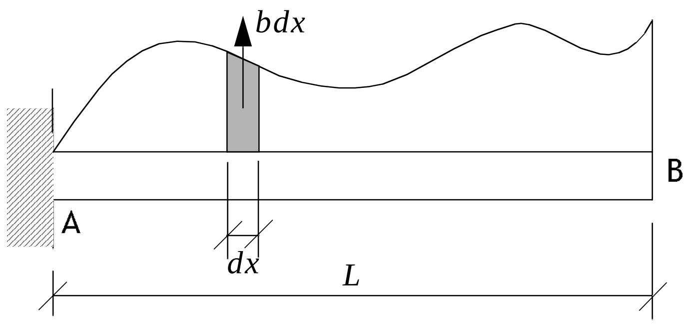
Concentrated Force



$$\beta_{B,o} = \frac{P^*a^2}{2D_B}$$

$$v_{B,o} = \frac{P^*a}{D_T} + \frac{P^*a^3}{3D_B} + \frac{P^*a^2}{2D_B}(L-a)$$

Distributed Loading



Replace P^* with bdx and integrate from $x = 0$ to $x = L$

$$\beta_{B,o} = \int_0^L \frac{x^2}{2D_B} b dx$$

$$v_{B,o} = \int_0^L \frac{x}{D_T} b dx + \int_0^L \frac{x^3}{3D_B} b dx + \int_0^L \frac{x^2}{2D_B} b dx (L-x)$$

for b constant (ie uniformly distributed loading)

$$\beta_{B,o} = \frac{bL^3}{6D_B}$$

$$v_{B,o} = \frac{bL^2}{2D_T} + \frac{bL^4}{8D_B}$$

1.9 Summary

$$u_B = u_{B,o} + \frac{\bar{F}_B L}{D_S} + u_A$$

$$v_B = v_{B,o} + \frac{\bar{M}_B}{D_B}\frac{L^2}{2} + \frac{\bar{V}_B}{D_B}\frac{L^3}{3} + \frac{\bar{V}_B}{D_T}L + v_A + \beta_A L$$

$$\beta_B = \beta_{B,o} + \frac{\bar{M}_B L}{D_B} + \frac{\bar{V}_B}{D_B}\frac{L^2}{2} + \beta_A$$

These equations can be written as

$$\begin{bmatrix} u_B \\ v_B \\ \beta_B \end{bmatrix} = \begin{bmatrix} u_{B,o} \\ v_{B,o} \\ \beta_{B,o} \end{bmatrix} + \begin{bmatrix} \frac{L}{D_S} & 0 & 0 \\ 0 & \frac{L^3}{3D_B} + \frac{L}{D_T} & \frac{L^2}{2D_B} \\ 0 & \frac{L^2}{2D_B} & \frac{1}{D_B} \end{bmatrix} \begin{bmatrix} \bar{F}_B \\ \bar{V}_B \\ \bar{M}_B \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & L \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_A \\ v_A \\ \beta_A \end{bmatrix}$$

Rigid body transformation from A to B

Also

$$\bar{F}_A = \bar{F}_{A,o} - \bar{F}_B$$

$$\bar{V}_A = \bar{V}_{A,o} - \bar{V}_B$$

$$\bar{M}_A = \bar{M}_{A,o} - \bar{M}_B - L\bar{V}_B$$

$$\begin{bmatrix} \bar{F}_A \\ \bar{V}_A \\ \bar{M}_A \end{bmatrix} = \begin{bmatrix} \bar{F}_{A,o} \\ \bar{V}_{A,o} \\ \bar{M}_{A,o} \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & L & 1 \end{bmatrix} \begin{bmatrix} \bar{F}_B \\ \bar{V}_B \\ \bar{M}_B \end{bmatrix}$$

1.10 Matrix Formulation - Straight Members

Define:

$$\underline{u}_B = \begin{bmatrix} u_B \\ v_B \\ \beta_B \end{bmatrix} \text{ Displacement Matrix}$$

$$\underline{F}_B = \begin{bmatrix} \bar{F}_B \\ \bar{V}_B \\ \bar{M}_B \end{bmatrix} \text{ End Action Matrix}$$

General Force Displacemnt Relation

Express displacement at B as:

$$\underline{u}_B = \underline{u}_{B,o} + \underline{f}_B \underline{F}_B + \underline{T}_{AB} \underline{u}_A$$

$$\left.\begin{array}{r} \underline{u}_{B,o}\text{: Due to applied loading} \\ \underline{f}_B \underline{F}_B\text{: Due to forces at B} \\ \underline{T}_{AB} \underline{u}_A\text{: Effect of motion at A} \end{array}\right\} \text{Based on cantilever model}$$

Interpret

$\underline{f}_B$ = Member flexibility matrix

$\underline{T}_{AB}$ = Rigid body transformation from A to B

For the prismatic case

$$\underline{f}_B = \begin{bmatrix} \frac{L}{D_S} & 0 & 0 \\ 0 & \frac{L^3}{3D_B} + \frac{L}{D_T} & \frac{L^2}{2D_B} \\ 0 & \frac{L^2}{2D_B} & \frac{1}{D_B} \end{bmatrix}$$

Force Displacement Relations

Define $\underline{k}_B = \underline{f}_B^{-1}$ = Member stiffness matrix

Start with

$$\underline{u}_B = \underline{u}_{B,o} + \underline{f}_B \underline{F}_B + \underline{T}_{AB} \underline{u}_A$$

Solve for $\underline{F}_B$

$$\underline{f}_B \underline{F}_B = \underline{u}_B - \underline{T}_{AB} \underline{u}_A - \underline{u}_{B,o}$$

$$\underline{F}_B = \underline{k}_B \underline{u}_B - \underline{k}_B \underline{T}_{AB} \underline{u}_A - \underline{k}_B \underline{u}_{B,o}$$

Define

$$\underline{F}_{B,i} = -\underline{k}_B \underline{u}_{B,o}$$

Then

$$\underline{F}_B = \underline{k}_B \underline{u}_B - \underline{k}_B \underline{T}_{AB} \underline{u}_A + \underline{F}_{B,i}$$

Next, determine $\underline{F}_A$

$$\underline{F}_A = \underline{F}_{A,o} - \underline{T}_{AB}^T \underline{F}_B$$

$$\underline{F}_A = (-\underline{T}_{AB}^T \underline{k}_B)\underline{u}_B + (\underline{T}_{AB}^T \underline{k}_B \underline{T}_{AB})\underline{u}_A + \underline{F}_{A,i}$$

where

$$\underline{F}_{A,i} = \underline{F}_{A,o} - \underline{T}_{AB}^T \underline{F}_{B,i}$$

Note $\underline{F}_{A,i}$ and $\underline{F}_{B,i}$ are the initial end actions with no end displacements

Finally, rewrite as

$$\underline{F}_B = \underline{k}_{BB} \underline{u}_B + \underline{k}_{BA} \underline{u}_A + \underline{F}_{B,i}$$

$$\underline{F}_A = \underline{k}_{BA}^T \underline{u}_B + \underline{k}_{AA} \underline{u}_A + \underline{F}_{A,i}$$

Notice that there are two fundamental matrices: $\underline{k}_B$ and $\underline{T}_{AB}$

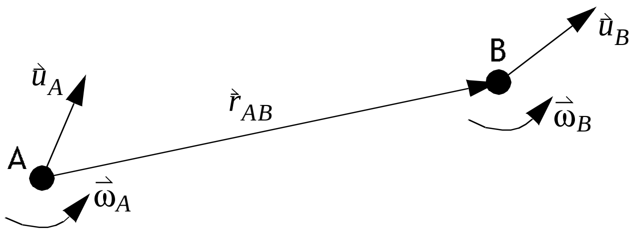
Matrices for Prismatic Case

	u_B	v_B	β_B	u_A	v_A	β_A
F_B	$\frac{D_S}{L}$	0	0	$-\frac{D_S}{L}$	0	0
V_B	0	$\frac{12D^*_B}{L^3}$	$\frac{(-6)D^*_B}{L^2}$	0	$\frac{(-12)D^*_B}{L^3}$	$\frac{(-6)D^*_B}{L^2}$
M_B	0	$\frac{(-6)D^*_B}{L^2}$	$\frac{(4+a)D^*_B}{L}$	0	$\frac{6D^*_B}{L^2}$	$\frac{(2+a)D^*_B}{L}$
F_A	$-\frac{D_S}{L}$	0	0	$\frac{D_S}{L}$	0	0
V_A	0	$\frac{(-12)D^*_B}{L^3}$	$\frac{6D^*_B}{L^2}$	0	$\frac{12D^*_B}{L^3}$	$\frac{6D^*_B}{L^2}$
M_A	0	$\frac{(-6)D^*_B}{L^2}$	$\frac{(2+a)D^*_B}{L}$	0	$\frac{6D^*_B}{L^2}$	$\frac{(4+a)D^*_B}{L}$

$$a = \frac{12D_B}{L^2 D_T} \qquad D^*_B = \frac{D_B}{(1+a)}$$

1.11 Transformation Relations

Rigid Body Displacement Transformation



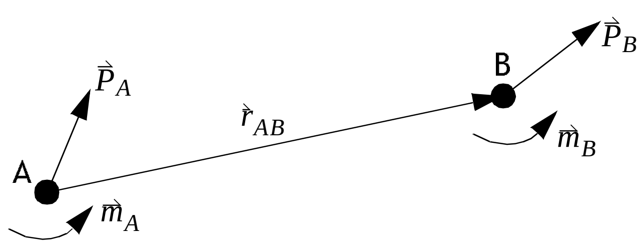
$$\vec{\omega}_B = \vec{\omega}_A$$
$$\vec{u}_B = \vec{u}_A + \vec{\omega}_A \times \vec{r}_{AB}$$

$$\left.\begin{aligned} u_B &= u_A \\ v_B &= v_A + \omega_A L \\ \omega_B &= \omega_A \end{aligned}\right\}\text{in two dimensions}$$

$$\begin{bmatrix} u_B \\ v_B \\ \omega_B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & L \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_A \\ v_A \\ \omega_A \end{bmatrix}$$
$$\vec{u}_B = T_{AB}\vec{u}_A$$

Statically Equivalent Force Transformation

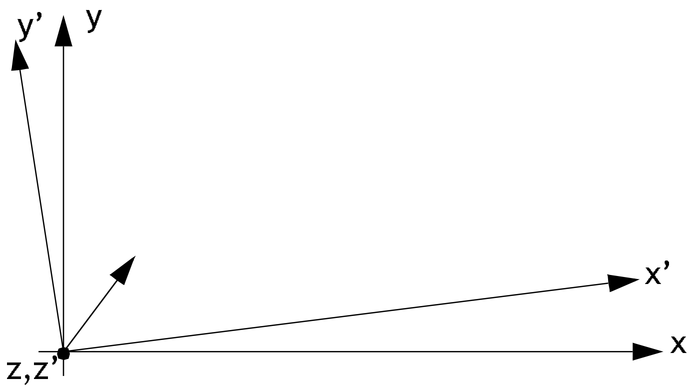
Translate force system acting at B to point A



$$\vec{P}_A = \vec{P}_B$$
$$\vec{m}_A = \vec{m}_B + \vec{r}_{AB} \times \vec{P}_B$$

$$\underline{F}_A = \underline{F}_{A,\,o} - \underline{T}_{AB}^T \underline{F}_B$$

Coordinate Transformation



$$\underline{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \qquad \underline{a}' = \begin{bmatrix} a_{x'} \\ a_{y'} \\ a_{z'} \end{bmatrix}$$

$$\underline{a}' = \underline{R}\,\underline{a}$$

$$\underline{R} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The inverse is

$$\left.\begin{aligned} \cos\theta &= \cos(-\theta) \\ \sin\theta &= -\sin(-\theta) \\ \underline{R}^{-1} &= \underline{R}(-\theta) \end{aligned}\right\} \; \underline{R}^T = \underline{R}^{-1}$$

Take

(x, y, z) = Global frame
(x', y', z') = Local frame

$$\underline{F}^{(l)} = \underline{R}^{(gl)}\underline{F}^{(g)}$$

$$\underline{R}^{(gl)} = \underline{R}$$

Given $\underline{k}$ in local frame ($\underline{k}^{(l)}$), transform to global frame

$$\underline{F}^{(l)} = \underline{k}^{(l)}\underline{u}^{(l)} = \underline{k}^{(l)}\underline{R}^{(gl)}\underline{u}^{(g)}$$

$$\underline{F}^{(g)} = \underline{R}^{(lg)}\underline{F}^{(l)} = \underline{R}^{(lg)}\underline{k}^{(l)}\underline{R}^{(gl)}\underline{u}^{(g)}$$

If

$$\underline{F}^{(g)} = \underline{k}^{(g)}\underline{u}^{(g)}$$

Then

$$\underline{k}^{(g)} = \underline{R}^{(lg)}\underline{k}^{(l)}\underline{R}^{(gl)} = (\underline{R}^{(gl)})^T\underline{k}^{(l)}\underline{R}^{(gl)}$$

1.12 Structural Stiffness Matrix assembly

$$\underline{F}_B^g = \underline{k}_{BB}^g\underline{u}_B^g + \underline{k}_{BA}^g\underline{u}_A^g + \underline{F}_{B,\,i}^g$$

$$\underline{F}_A^g = (\underline{k}_{BA}^g)^T\underline{u}_B^g + \underline{k}_{AA}^g\underline{u}_A^g + \underline{F}_{A,\,i}^g$$

$$\underline{F}_i^g = \underline{R}^T\underline{F}_i^l$$

Use direct stiffness method to generate the system equations referred to the global frame.

Take B as the positive end and A as the negative end.

$\mathrm{B} \to n+$

$\mathrm{A} \to n-$

for member n

Write system equation as

$$\underline{\ }_E = \underline{P}_I + \underline{K}\,\underline{U}$$

Work with the partitioned form of system stiffness matrix $\underline{K}$.

$\underline{k}_{BB}$ in n+,n+

$\underline{k}_{AA}$ in n-,n-

$\underline{k}_{BA}^T$ in n+,n-

$\underline{k}_{BA}^T$ in n-,n+

with

$\underline{F}_{B,\,i}$ in n+ of $\underline{P}_I$

$\underline{F}_{A,\,i}$ in n- of $\underline{P}_I$