16

Direct Stiffness Method —Linear System

16–1. INTRODUCTION

We consider a system comprised of m members which are connected at j joints. We suppose the geometry of the assembled system is defined with respect to a global frame† and use a superscript o to indicate quantities referred to the global frame. The external force and displacement matrices for joint k are denoted by $\mathscr{P}_k^o, \mathscr{U}_k^o$:

$$\underset{(i\times 1)}{\mathscr{P}_k^o} = \begin{Bmatrix} \mathbf{P}_k^o \\ \hline \mathbf{T}_k^o \end{Bmatrix} \begin{matrix} (\alpha\times 1) \\ (\beta\times 1) \end{matrix}$$

$$\underset{(i\times 1)}{\mathscr{U}_k^o} = \begin{Bmatrix} \mathbf{u}_k^o \\ \hline \boldsymbol{\omega}_k^o \end{Bmatrix} \begin{matrix} (\alpha\times 1) \\ (\beta\times 1) \end{matrix} \qquad (16\text{–}1)$$

where α is the number of translation (force) components, β is the number of rotation (moment) components, and $i = \alpha + \beta$. Note that $\alpha = 2, \beta = 1$ for a planar system subjected to in-plane loading and $\alpha = 1, \beta = 2$ for a planar system subjected to out-of-plane loading. For an arbitrary system, $\alpha = \beta = 3$.

In what follows, we assume the material is linearly elastic and the geometry is linear, i.e., we neglect the change in geometry due to deformation. The governing equations consist of joint force-equilibrium equations and member force-displacement relations. We have already developed the member force-displacement relations in Chapter 15, so that it remains only to establish the joint force-equilibrium equations. In this chapter, we apply the direct stiffness method, which consists in assembling the system stiffness and initial force matrices by superimposing the contribution of each member. In the next chapter, we present the general formulation for a linear member system and obtain the equations corresponding to the force and displacement solution by

† By *global* frame, we mean a fixed cartesian frame.

matrix operations. Finally, in Chapter 18, we extend the direct stiffness method to include geometrical nonlinearity.

16–2. MEMBER FORCE-DISPLACEMENT RELATIONS

In developing the relations between the end forces and end displacements for a member, we considered the member geometry and loading to be referred to a basic member frame (frame n) and used A, B to denote the negative and positive ends of the member. The general relations were written (see (15–107)) as

$$\begin{aligned} \mathscr{F}^n_B &= \mathscr{F}^n_{B,i} + \mathbf{k}^n_{BB}\mathscr{U}^n_B + \mathbf{k}^n_{BA}\mathscr{U}^n_A \\ \mathscr{F}^n_A &= \mathscr{F}^n_{A,i} + \mathbf{k}^{n,T}_{BA}\mathscr{U}^n_B + \mathbf{k}^n_{AA}\mathscr{U}^n_A \end{aligned} \tag{a}$$

Note that (a) also applies when there is only partial end restraint or internal releases.

Now, we define n_+, n_- as the joints at the positive and negative ends of member n. Replacing B by n_+, A by n_-, the force-displacement relations for member n referred to the *member* frame take the form

$$\begin{aligned} \mathscr{F}^n_{n_+} &= \mathscr{F}^n_{n_+,i} + \mathbf{k}^n_{n_+n_+}\mathscr{U}^n_{n_+} + \mathbf{k}^n_{n_+n_-}\mathscr{U}^n_{n_-} \\ \mathscr{F}^n_{n_-} &= \mathscr{F}^n_{n_-,i} + \mathbf{k}^n_{n_-n_+}\mathscr{U}^n_{n_+} + \mathbf{k}^n_{n_-n_-}\mathscr{U}^n_{n_-} \end{aligned} \tag{b}$$

where

$$\mathbf{k}^n_{n_-n_+} = (\mathbf{k}^n_{n_+n_-})^T \tag{c}$$

We transform the force and displacement quantities from the member frame to the global frame for the system by applying

$$\begin{aligned} \mathscr{U}^n &= \mathscr{R}^{on}\mathscr{U}^o \\ \mathscr{F}^o &= \mathscr{R}^{on,T}\mathscr{F}^n \end{aligned} \tag{16–2}$$

to (b). This step is necessary since we are working with *joint* forces and displacements referred to the global frame. The final expressions are:

$$\begin{aligned} \mathscr{F}^o_{n_+} &= \mathscr{F}^o_{n_+,i} + \mathbf{k}^o_{n_+n_+}\mathscr{U}^o_{n_+} + \mathbf{k}^o_{n_+n_-}\mathscr{U}^o_{n_-} \\ \mathscr{F}^o_{n_-} &= \mathscr{F}^o_{n_-,i} + \mathbf{k}^o_{n_-n_+}\mathscr{U}^o_{n_+} + \mathbf{k}^o_{n_-n_-}\mathscr{U}^o_{n_-} \end{aligned} \tag{16–3}$$

where the global member stiffness and initial force matrices are generated with

$$\begin{aligned} \mathbf{k}^o_{()()} &= \mathscr{R}^{on,T}\mathbf{k}^n_{()()}\mathscr{R}^{on} \\ \mathscr{F}^o_{(),i} &= \mathscr{R}^{on,T}\mathscr{F}^n_{(),i} \end{aligned} \tag{16–4}$$

Once the displacements are known, we evaluate $\mathscr{F}^o_{()}$ using (16–3) and then transform to the member frame.

Since the initial end force and stiffness matrices are generated in partitioned form, it is natural to express (16–4) in partitioned form. Using the notation

introduced in Section 15–8, we write

$$\begin{aligned} \mathscr{R}^{on} &= \begin{bmatrix} \underset{(\alpha\times\alpha)}{\mathbf{R}^{on}_\alpha} & \mathbf{0} \\ \mathbf{0} & \underset{(\beta\times\beta)}{\mathbf{R}^{on}_\beta} \end{bmatrix} \\ \bar{\mathscr{F}}_{()} &= \begin{Bmatrix} \bar{\mathbf{F}}_{()} \\ \bar{\mathbf{M}}_{()} \end{Bmatrix} \begin{matrix} (\alpha\times 1) \\ (\beta\times 1) \end{matrix} \\ \mathbf{k}_{()()} &= \begin{bmatrix} \underset{(\alpha\times\alpha)}{\mathbf{k}_{()(),11}} & \underset{(\alpha\times\beta)}{\mathbf{k}_{()(),12}} \\ \underset{(\beta\times\alpha)}{\mathbf{k}_{()(),21}} & \underset{(\beta\times\beta)}{\mathbf{k}_{()(),22}} \end{bmatrix} \end{aligned} \tag{16–5}$$

Expanding (16–4) leads to

$$\begin{aligned} \mathbf{k}^o_{()(),11} &= \mathbf{R}^{on,T}_\alpha\mathbf{k}^n_{()(),11}\mathbf{R}^{on}_\alpha \\ \mathbf{k}^o_{()(),12} &= \mathbf{R}^{on,T}_\alpha\mathbf{k}^n_{()(),12}\mathbf{R}^{on}_\beta \\ \mathbf{k}^o_{()(),21} &= \mathbf{R}^{on,T}_\beta\mathbf{k}^n_{()(),21}\mathbf{R}^{on}_\alpha = \mathbf{k}^{o,T}_{()(),12} \\ \mathbf{k}^o_{()(),22} &= \mathbf{R}^{on,T}_\beta\mathbf{k}^n_{()(),22}\mathbf{R}^{on}_\beta \\ \bar{\mathbf{F}}^o_{(),i} &= \mathbf{R}^{on,T}_\alpha\bar{\mathbf{F}}^n_{(),i} \\ \bar{\mathbf{M}}^o_{(),i} &= \mathbf{R}^{on,T}_\beta\bar{\mathbf{M}}^n_{(),i} \end{aligned} \tag{16–6}$$

Note that $\mathbf{k}^n_{()()}$ is a natural property of the member whereas $\mathbf{k}^o_{()()}$ depends on the orientation of the member frame with respect to the global frame. The operations defined by (16–6) can be considered as the element matrix generation phase.

The member force-displacement relations satisfy the equilibrium conditions for the member and compatibility between the restrained end displacements and the corresponding joint displacements. Actually, the equilibrium conditions were used to determine $\bar{\mathscr{F}}^n_A$. Compatibility is satisfied by setting $\mathscr{U}_B = \mathscr{U}_{n_+}$ and $\mathscr{U}_A = \mathscr{U}_{n_-}$. When there is only partial restraint at an end, there will be displacement discontinuities. For example, if there is a rotation release at the positive end, $\omega^o_{n_+}$ will not be equal to the end rotation matrix. We have treated† partial end restraint by defining an effective member stiffness matrix $\mathbf{k}_e$. In the derivation of $\mathbf{k}_e$, we consider $\mathscr{U}_B$, $\mathscr{U}_A$ to be the displacements of the *supports* (i.e., the joints) and enforce continuity of only the *restrained* end displacements.

16–3. SYSTEM EQUILIBRIUM EQUATIONS

The equilibrium equations for joint k are obtained by summing the end forces for the members incident on k:

$$\mathscr{P}^o_k = \sum_{r_+=k}\bar{\mathscr{F}}^o_{r_+} + \sum_{t_-=k}\bar{\mathscr{F}}^o_{t_-} \tag{a}$$

† See Sec. 16–12.

In general, $\mathscr{P}_k^o$ depends on $\mathscr{U}_k^o$ and the displacements of those joints which are connected to joint k. We define $\mathscr{P}$, $\mathscr{U}$ as the total (or system) external joint force and joint displacement matrices:

$$\begin{aligned} \mathscr{P} &= \{\mathscr{P}_1^o, \mathscr{P}_2^o, \ldots, \mathscr{P}_j^o\} \qquad (ij \times 1) \\ \mathscr{U} &= \{\mathscr{U}_1^o, \mathscr{U}_2^o, \ldots, \mathscr{U}_j^o\} \qquad (ij \times 1) \end{aligned} \tag{16–7}$$

and write the complete set of ij joint force-equilibrium equations as

$$\mathscr{P} = \mathscr{P}_o + \mathscr{K}\mathscr{U} \tag{16–8}$$

where $\mathscr{P}_o$ contains the joint forces required to equilibrate the initial end forces We have dropped the reference frame superscript for convenience.

The most efficient way to assemble $\mathscr{K}$ and $\mathscr{P}_o$ is to work with submatrices of order i, the *natural* partition size, and superimpose the contributions of each member which follow directly from (16–3). This operation requires no matrix multiplications. The terms due to member n are listed below.

In $\mathscr{P}_o$ (Partitioned Form Is $j \times 1$):

$$\begin{aligned} &\bar{\mathscr{F}}^o_{n_+,i} \qquad \text{in row } n_+ \\ &\bar{\mathscr{F}}^o_{n_-,i} \qquad \text{in row } n_- \end{aligned} \tag{16–9}$$

In $\mathscr{K}$ (Partitioned Form Is $j \times j$):

$$\begin{aligned} &\mathbf{k}^o_{n_+n_+} \qquad \text{in row } n_+, \text{ column } n_+ \\ &\mathbf{k}^o_{n_+n_-} \qquad \text{in row } n_+, \text{ column } n_- \\ &\mathbf{k}^{o,T}_{n_+n_-} \qquad \text{in row } n_-, \text{ column } n_+ \\ &\mathbf{k}^o_{n_-n_-} \qquad \text{in row } n_-, \text{ column } n_- \end{aligned} \tag{16–10}$$

Since $\mathscr{K}$ is symmetrical, only the upper or lower half has to be stored.

16–4. INTRODUCTION OF JOINT DISPLACEMENT RESTRAINTS

In this section, we extend the procedure described in Sec. 8–3 for introducing joint translation restraints in the formulation for an ideal truss to an arbitrary member system. Actually, only the notation for the joint force and joint displacement matrices has to be changed.

The governing equations are:

$$\mathscr{K}\mathscr{U} = \mathscr{P} - \mathscr{P}_o = \mathscr{P}_N$$

$$\begin{bmatrix} \mathscr{K}_{11} & \mathscr{K}_{12} & \cdots & \mathscr{K}_{1j} \\ \mathscr{K}_{12}^T & \mathscr{K}_{22} & \cdots & \mathscr{K}_{2j} \\ \vdots & \vdots & & \vdots \\ \mathscr{K}_{1j}^T & \mathscr{K}_{2j}^T & \cdots & \mathscr{K}_{jj} \end{bmatrix} \begin{Bmatrix} \mathscr{U}_1^o \\ \mathscr{U}_2^o \\ \vdots \\ \mathscr{U}_j^o \end{Bmatrix} = \begin{Bmatrix} -\mathscr{P}^o_{o,1} + \mathscr{P}_1^o \\ -\mathscr{P}^o_{o,2} + \mathscr{P}_2^o \\ \vdots \\ -\mathscr{P}^o_{o,j} + \mathscr{P}_j^o \end{Bmatrix} = \begin{Bmatrix} \mathscr{P}_{N,1} \\ \mathscr{P}_{N,2} \\ \vdots \\ \mathscr{P}_{N,j} \end{Bmatrix} \tag{16–11}$$

The stiffness and initial force matrices are assembled using (16–9) and (16–10). It remains to introduce the prescribed external forces and displacement restraints. If joint q is unrestrained, $\mathscr{P}_q^o$ is prescribed, and we just add $\mathscr{P}_q^o$ to $-\mathscr{P}^o_{o,q}$. If joint q is completely restrained, $\mathscr{P}_q^o$ is unknown. We replace the matrix equation for $\mathscr{P}_q^o$ with the matrix identity,

$$\mathscr{U}_q^o = \bar{\mathscr{U}}_q^o \tag{a}$$

Finally, if joint q is partially restrained, some of the elements in $\mathscr{P}_q$ are unknown. In this case, we replace the scalar equations for the unknown reactions by scalar identities.

We suppose joint q is partially restrained and, for generality, consider the translation and rotation restraint directions to be arbitrarily orientated with respect to the basic frame. We define $X_1', \ldots, X_\alpha'$ as the orthogonal directions for the translational restraint frame and $X_1'', \ldots, X_\beta''$ as the orthogonal directions for the rotational restraint frame. Quantities referred to the restraint frames are indicated with primes and a single superscript is used for the total matrix:

$$\begin{aligned} \underset{(i \times 1)}{\mathscr{U}_q^q} &= \begin{Bmatrix} \mathbf{u}_q' \\ \boldsymbol{\omega}_q'' \end{Bmatrix} \begin{matrix} (\alpha \times 1) \\ (\beta \times 1) \end{matrix} \\ \mathscr{P}_q^q &= \begin{Bmatrix} \mathbf{P}_q' \\ \mathbf{T}_q'' \end{Bmatrix} \end{aligned} \tag{16–12}$$

Now,

$$\begin{aligned} \mathbf{P}_q' &= \mathbf{R}_\alpha^{oq'} \mathbf{P}_q^o \\ \mathbf{T}_q'' &= \mathbf{R}_\beta^{oq''} \mathbf{T}_q^o \end{aligned} \tag{16–13}$$

We define $\mathscr{R}^{oq}$ as the total rotation transformation matrix:

$$\mathscr{R}^{oq} = \left[\begin{array}{c|c} \mathbf{R}_\alpha^{oq'} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{R}_\beta^{oq''} \end{array}\right] \tag{16–14}$$

With this notation, the transformation laws take the form

$$\begin{aligned} \mathscr{P}_q^q &= \mathscr{R}^{oq} \mathscr{P}_q^o \\ \mathscr{U}_q^o &= \mathscr{R}^{oq,T} \mathscr{U}_q^q \end{aligned} \tag{16–15}$$

The modification requires two operations. First, we transform $\mathscr{P}_q^o$, $\mathscr{U}_q^o$ in (16–11) to $\mathscr{P}_q^q$, $\mathscr{U}_q^q$. This is accomplished by premultiplying row q of $\mathscr{K}$, $\mathscr{P}_o$ with $\mathscr{R}^{oq}$ and postmultiplying column q of $\mathscr{K}$ with $\mathscr{R}^{oq,T}$. In the second step, we replace the equations corresponding to the unknown elements in $\mathscr{P}_q^q$ with identities. This operation can also be represented in matrix form.

Suppose the rth element in $\mathscr{U}_q^q$ is prescribed. We assemble four matrices, $\mathbf{E}_q$, $\mathbf{G}_q$, $\mathscr{U}_q^*$, and $\mathscr{P}_q^*$, as follows:

1. $\mathbf{E}_q$ *and* $\mathbf{G}_q$

We start with

$$\mathbf{E} = \mathbf{I}_i \qquad \mathbf{G} = \mathbf{O}_i$$

and set

$$E_{rr} = 0 \qquad G_{rr} = +1$$

2. $\mathscr{U}_q^*$

We start with an ith-order column matrix having zero elements and set the element in row r equal to the prescribed displacement.

3. $\mathscr{P}_q^*$

We start with an ith-order column matrix having zero elements and enter the values of the prescribed forces and moments referred to the restraint frames. Note that element r is zero.

Premultiplying transformed row q of $\mathscr{K}$, $\mathscr{P}_o$ with $\mathbf{E}_q$ reduces the rth equation to $0 = 0$. Then, adding $\mathbf{G}_q$ to $\mathbf{E}_q\mathscr{K}_{qq}$ and $\mathscr{P}_q^* + \mathscr{U}_q^*$ to $-\mathbf{E}_q\mathscr{P}_{o,q}$ introduces the identity for the rth element of $\bar{\mathscr{U}}_q^o$ and includes the prescribed external forces in $\mathscr{P}_{N,q}$. We also operate on the qth column of $\mathscr{K}$ to *preserve symmetry* and include the terms due to prescribed displacements in $\mathscr{P}_N$. The complete set of operations for joint q are listed below:

1. $\ell = 1, 2, \ldots, q-1$

$$\begin{aligned} \mathscr{P}_{N,\ell} &= \mathscr{P}_{N,\ell} - (\mathscr{K}_{\ell q}\mathscr{R}^{oq,T})\mathscr{U}_q^* \\ \mathscr{K}_{\ell q} &= (\mathscr{K}_{\ell q}\mathscr{R}^{oq,T})\mathbf{E}_q \end{aligned}$$

$$\begin{aligned} \text{2.}\quad \mathscr{P}_{N,q} &= \mathbf{E}_q[\mathscr{R}^{oq}\mathscr{P}_{N,q} - (\mathscr{R}^{oq}\mathscr{K}_{qq}\mathscr{R}^{oq,T})\mathscr{U}_q^*] + \mathscr{U}_q^* + \mathscr{P}_q^* \\ \mathscr{K}_{qq} &= \mathbf{E}_q(\mathscr{R}^{oq}\mathscr{K}_{qq}\mathscr{R}^{oq,T})\mathbf{E}_q + \mathbf{G}_q \end{aligned} \tag{16-16}$$

3. $\ell = q+1, q+2, \ldots, j$

$$\begin{aligned} \mathscr{K}_{q\ell} &= \mathbf{E}_q(\mathscr{R}^{oq}\mathscr{K}_{q\ell}) \\ \mathscr{P}_{N,\ell} &= \mathscr{P}_{N,\ell} - (\mathscr{K}_{q\ell}^T\mathscr{R}^{oq,T})\mathscr{U}_q^* \end{aligned}$$

The operations defined by (16–16) are carried out for each joint, working with successive joint members. We represent the modified equations as

$$\mathscr{K}^*\mathscr{U}^J = \mathscr{P}_N^* \tag{16-17}$$

The superscript J is placed on $\mathscr{U}$ to indicate that the joint displacement matrices are referred to the *local* joint restraint frames, which may not coincide with the global frame. Again we point out that the primary advantage of this modification procedure is that *no* row or column rearrangement is required. Solving (16–17) yields the joint displacements (local restraint frame) listed in their natural order, i.e., according to increasing joint number. The modified stiffness matrix, $\mathscr{K}^*$, will be *positive definite* when the system is stable.

Once $\mathscr{U}^J$ is known, we transform the displacements from the restraint frames to the global frame, using (16–15), and evaluate the member end forces from (16–3). Next, we assemble the total external force matrix, $\mathscr{P}$. The contribution

of member n is

$$\begin{aligned} &\bar{\mathscr{F}}_{n+}^o \quad \text{in row } n_+ \\ &\bar{\mathscr{F}}_{n-}^o \quad \text{in row } n_- \end{aligned} \tag{16-18}$$

Finally, we transform the external joint forces from the global frame to the local restraint frames. This step determines the reactions and also provides a statics check on the solution.

Example 16–1

Suppose joint q is completely restrained. Then, $\mathscr{U}_q^* = \bar{\mathscr{U}}_q^o$ and $\mathscr{P}_q^* = \mathbf{0}$. The forms for $\mathbf{E}$, $\mathbf{G}$ are

$$\mathbf{E}_q = \mathbf{0}_i \qquad \mathbf{G}_q = \mathbf{I}_i \tag{a}$$

and (16–16) reduces to

1. $\ell = 1, 2, \ldots, q-1$

$$\begin{aligned} \mathscr{P}_{N,\ell} &= \mathscr{P}_{N,\ell} - \mathscr{K}_{\ell q}\bar{\mathscr{U}}_q^o \\ \mathscr{K}_{\ell q} &= \mathbf{0}_i \end{aligned}$$

2.

$$\begin{aligned} \mathscr{P}_{N,q} &= \bar{\mathscr{U}}_q^o \\ \mathscr{K}_{qq} &= \mathbf{I}_i \end{aligned} \tag{b}$$

3. $\ell = q+1, q+2, \ldots, j$

$$\begin{aligned} \mathscr{K}_{q\ell} &= \mathbf{0}_i \\ \mathscr{P}_{N,\ell} &= \mathscr{P}_{N,\ell} - \mathscr{K}_{q\ell}^T\bar{\mathscr{U}}_q^o \end{aligned}$$

Example 16–2

Suppose joint q is completely restrained against translation. Then, the translation matrix $\bar{\mathbf{u}}_q^o$ and external moment matrix $\mathbf{T}_q^o$ are prescribed. The appropriate matrices for this case are

$$\begin{aligned} \mathbf{E}_q &= \left[\begin{array}{c|c} \mathbf{0}_\alpha & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I}_\beta \end{array}\right] \qquad \mathbf{G}_q = \left[\begin{array}{c|c} \mathbf{I}_\alpha & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0}_\beta \end{array}\right] \\ \mathscr{U}_q^* &= \begin{Bmatrix} \bar{\mathbf{u}}_q^o \\ \mathbf{0}_\beta \end{Bmatrix} \qquad \mathscr{P}_q^* = \begin{Bmatrix} \mathbf{0}_\alpha \\ \mathbf{T}_q^o \end{Bmatrix} \\ \mathscr{R}^{oq} &= \mathbf{I}_i \end{aligned} \tag{a}$$

Example 16–3

We consider the case where joint q is restrained with respect to translation in *one* direction and there is no restraint against rotation. This corresponds to a "roller" support. We take X_1' to coincide with the *restraint* direction and X_2', X_1' as mutually orthogonal directions comprising a right-handed system. The translation, u'_{q1}, is prescribed. The prescribed forces are P'_{q2}, P'_{q3}, and $\mathbf{T}_q^o$.

We first assemble $\mathcal{R}^{oq}$. From (16–14),

$$\mathcal{R}^{oq} = \left[\begin{array}{c|c} \mathbf{R}_3^{oq'} & \\ \hline & \mathbf{I}_3 \end{array}\right] \tag{a}$$

where

$$\begin{aligned} \mathbf{R}_3^{oq'} &= [\ell_{rs}] \qquad r, s = 1, 2, 3 \\ \ell_{rs} &= \cos(X'_r, X^o_s) \end{aligned} \tag{b}$$

The forms of $\mathbf{E}$, $\mathbf{G}$, $\mathcal{U}^*$, and $\mathcal{P}^*$ are

$$\mathbf{E}_q = \left[\begin{array}{ccc|c} 0 & & & \\ & 1 & & \\ & & 1 & \\ \hline & & & \mathbf{I}_3 \end{array}\right] \qquad \mathbf{G}_q = \left[\begin{array}{ccc|c} 1 & & & \\ & 0 & & \\ & & 0 & \\ \hline & & & \mathbf{0}_3 \end{array}\right]$$

$$\mathcal{U}^*_q = \left\{\begin{array}{c} \bar{u}'_{q1} \\ 0 \\ 0 \\ \hline \mathbf{0}_3 \end{array}\right\} \qquad \mathcal{P}^*_q = \left\{\begin{array}{c} 0 \\ \bar{P}'_{q2} \\ \bar{P}'_{q3} \\ \hline \bar{\mathbf{T}}^o_q \end{array}\right\}. \tag{c}$$

We specialize the results for a planar system subjected to planar loading. In order for only planar deformation to occur, the translation restraint direction must lie in the plane of the system, which we take as the X^o_1-X^o_2 plane. It is convenient to select the orientation of X'_2 such that X'_3 coincides with X^o_3. The specialized forms are

$$\mathcal{R}^{oq} = \left[\begin{array}{c|c} \mathbf{R}_2^{oq'} & \\ \hline & 1 \end{array}\right]$$

$$\begin{aligned} \mathbf{R}_2^{oq'} &= [\ell_{rs}] \qquad r, s = 1, 2 \\ \ell_{rs} &= \cos(X'_r, X^o_s) \end{aligned}$$

$$\mathbf{E}_q = \left[\begin{array}{cc|c} 0 & & \\ & 1 & \\ \hline & & 1 \end{array}\right] \qquad G_q = \left[\begin{array}{cc|c} 1 & & \\ & 0 & \\ \hline & & 0 \end{array}\right] \tag{d}$$

$$\mathcal{U}^*_q = \left\{\begin{array}{c} \bar{u}'_{q1} \\ 0 \\ \hline 0 \end{array}\right\} \qquad \mathcal{P}^*_q = \left\{\begin{array}{c} 0 \\ \bar{P}'_{q2} \\ \hline \bar{T}^o_{q3} \end{array}\right\}$$

Finally, we consider the case of a planar system subjected to an out-of-plane loading. The translational restraint direction must be parallel to the X^o_3 direction in order for only out-of-plane deformation to occur. For this case, u^o_{q3}, T^o_{q1}, and T^o_{q2} are prescribed. The specialized forms are

$$\mathcal{R}^{oq} = \mathbf{I}_3$$

$$\mathbf{E}_q = \left[\begin{array}{c|c} 0 & \\ \hline & \mathbf{I}_2 \end{array}\right] \qquad \mathbf{G}_q = \left[\begin{array}{c|c} 1 & \\ \hline & \mathbf{0}_2 \end{array}\right]$$

$$\mathcal{U}^*_q = \left\{\begin{array}{c} \bar{u}^o_{q3} \\ \hline \mathbf{0}_2 \end{array}\right\} \qquad \mathcal{P}^*_q = \left\{\begin{array}{c} 0 \\ \hline \bar{T}^o_{q1} \\ \bar{T}^o_{q2} \end{array}\right\} \tag{e}$$

Note that (e) is obtained by setting $\alpha = 1$, $\beta = 2$ in (a) of Example 16–2.

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