9

Force Method—Ideal Truss

9–1. GENERAL

The basic equations for the *linear geometric* case have the form

$$\bar{\mathbf{P}}_1 = \mathbf{B}_1\mathbf{F} \tag{a}$$

$$\mathbf{e} = \mathbf{B}_1^T\mathbf{U}_1 + \mathbf{B}_2^T\bar{\mathbf{U}}_2 = \mathbf{e}_0 + \mathbf{f}\mathbf{F} \tag{b}$$

$$\mathbf{P}_2 = \mathbf{B}_2\mathbf{F} \tag{c}$$

where the elements of $\mathbf{B}_1$ and $\mathbf{B}_2$ are constants. Equation (a) represents n_d linear equations relating the n_d prescribed joint forces and the m unknown bar forces. For the system to be initially stable, $r(\mathbf{B}_1) = n_d$, that is, the rows of $\mathbf{B}_1$ must be linearly independent. This requires $m \geqslant n_d$. In what follows, we consider only stable systems. If $m = n_d$, the system is said to be statically determinate since one can find the bar forces and reactions using only the equations of statics. The defect of (a) is equal to $m - n_d = q$, and is called the *degree of indeterminacy*. One can solve (a) for n_d bar forces in terms of the applied forces and q bar forces. We refer to the system defined by the n_d bars as the primary structure and the q unknown forces as force redundants. In order to determine $\mathbf{F}$, q additional equations relating the bar forces are required. These equations are called compatibility conditions and are obtained by operating on (b) which represents m relations between the n_d unknown displacements and the bar forces.

The general procedure outlined above is called the *force* or *flexibility* method. This procedure is applicable only when the geometry is *linear*. In what follows, we first develop the governing equations for the force method by operating on (a)–(c). We then show how one can establish the compatibility equations using the principle of virtual forces and discuss the extremal character of the force redundants. Finally, we compare the force method for a truss with the mesh method for an electrical network.

9–2. GOVERNING EQUATIONS—ALGEBRAIC APPROACH

We consider the first n_d columns of $\mathbf{B}_1$ to be linearly independent (if the system is initially stable, one can always renumber the bars such that this condition is satisfied) and partition $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{F}$ as follows:

$$\begin{aligned}
\underset{(n_d \times m)}{\mathbf{B}_1} &= \left[\underset{(n_d \times n_d)}{\mathbf{B}_{11}} \;\middle|\; \underset{(n_d \times q)}{\mathbf{B}_{12}}\right] \\
\underset{(r \times m)}{\mathbf{B}_2} &= \left[\underset{(r \times n_d)}{\mathbf{B}_{21}} \;\middle|\; \underset{(r \times q)}{\mathbf{B}_{22}}\right] \\
\overset{(m \times 1)}{\mathbf{F}} &= \left\{\begin{array}{c} \overset{(n_d \times 1)}{\mathbf{F}_1} \\ \hline \underset{(q \times 1)}{\mathbf{F}_2} \end{array}\right\}
\end{aligned} \tag{9–1}$$

The bars corresponding to $\mathbf{F}_1$ comprise the *primary structure* and $\mathbf{F}_2$ contains the q redundant bar forces. Using (9–1), the force-equilibrium equations ((a) and (c)) take the form

$$\mathbf{B}_{11}\mathbf{F}_1 = \bar{\mathbf{P}}_1 - \mathbf{B}_{12}\mathbf{F}_2 \qquad (n_d \text{ eqs.}) \tag{9–2}$$

$$\mathbf{P}_2 = \mathbf{B}_{21}\mathbf{F}_1 + \mathbf{B}_{22}\mathbf{F}_2 \qquad (r \text{ eqs.}) \tag{9–3}$$

Since $|\mathbf{B}_{11}| \neq 0$, we can solve (9–2) for $\mathbf{F}_1$, considering $\bar{\mathbf{P}}_1$, $-\mathbf{B}_{12}$ as right-hand sides. The complete set of $q + 1$ solutions is written as

$$\mathbf{F}_1 = \mathbf{F}_{1,0} + \mathbf{F}_{1,F_2}\mathbf{F}_2 \tag{9–4}$$

where $\mathbf{F}_{1,0}$ and $\mathbf{F}_{1,F_2}$ satisfy

$$\begin{aligned}
\mathbf{B}_{11}\mathbf{F}_{1,0} &= \bar{\mathbf{P}}_1 \\
\mathbf{B}_{11}F_{1,F_2} &= -\mathbf{B}_{12}
\end{aligned} \tag{9–5}$$

Note that the kth column of $\mathbf{F}_{1,F_2}$ contains the bar forces in the primary structure due to a unit value of the kth element in $\mathbf{F}_2$. Also, $\mathbf{F}_{1,0}$ contains the bar forces in the primary structure due to the applied joint loads, $\bar{\mathbf{P}}_1$, with $\mathbf{F}_2 = 0$. The reactions follow from (9–3):

$$\begin{aligned}
\mathbf{P}_2 &= \mathbf{P}_{2,0} + \mathbf{P}_{2,F_2}\mathbf{F}_2 \\
\mathbf{P}_{2,0} &= \mathbf{B}_{21}\mathbf{F}_{1,0} \\
\mathbf{P}_{2,F_2} &= \mathbf{B}_{21}\mathbf{F}_{1,F_2} + \mathbf{B}_{22}
\end{aligned} \tag{9–6}$$

We consider next (b). Partitioning $\mathbf{e}$, $\mathbf{e}_0$, and $\mathbf{f}$,

$$\overset{(m \times 1)}{\mathbf{e}} = \left\{\begin{array}{c} \overset{(n_d \times 1)}{\mathbf{e}_1} \\ \hline \underset{(q \times 1)}{\mathbf{e}_2} \end{array}\right\} \qquad \mathbf{e}_0 = \left\{\begin{array}{c} \mathbf{e}_{1,0} \\ \hline \mathbf{e}_{2,0} \end{array}\right\} \tag{9–7}$$

$$\mathbf{f} = \left[\begin{array}{c|c} \overset{(n_d \times n_d)}{\mathbf{f}_1} & 0 \\ \hline 0 & \underset{(q \times q)}{\mathbf{f}_2} \end{array}\right]$$

and using (9–1), the force-displacement relations expand to

$$\mathbf{B}_{11}^T\mathbf{U}_1 + \mathbf{B}_{21}^T\bar{\mathbf{U}}_2 = \mathbf{e}_1 = \mathbf{e}_{1,0} + \mathbf{f}_1\mathbf{F}_1 \qquad (n_d \text{ eqs.}) \qquad (9\text{–}8)$$
$$\mathbf{B}_{12}^T\mathbf{U}_1 + \mathbf{B}_{22}^T\bar{\mathbf{U}}_2 = \mathbf{e}_2 = \mathbf{e}_{2,0} + \mathbf{f}_2\mathbf{F}_2 \qquad (q \text{ eqs.}) \qquad (9\text{–}9)$$

Once $\mathbf{e}_1$ is known, (9–8) can be solved for $\mathbf{U}_1$.

We obtain the equation for $\mathbf{F}_2$ by eliminating $\mathbf{U}_1$ in (9–9). First (see (9–5)) we note that

$$\mathbf{B}_{12}^T = -(\mathbf{B}_{11}\mathbf{F}_{1,F_2})^T = -\mathbf{F}_{1,F_2}^T\mathbf{B}_{11}^T \qquad \text{(a)}$$

Then, premultiplying (9–8) by $\mathbf{F}_{1,F_2}^T$, adding the result to (9–9), and using (a), (9–6) leads to

$$\mathbf{P}_{2,F_2}^T\bar{\mathbf{U}}_2 = \mathbf{e}_2 + \mathbf{F}_{1,F_2}^T\mathbf{e}_1 \qquad (9\text{–}10)$$
$$= \mathbf{e}_{2,0} + \mathbf{f}_2\mathbf{F}_2 + \mathbf{F}_{1,F_2}^T(\mathbf{e}_{1,0} + \mathbf{f}_1\mathbf{F}_1) \qquad (9\text{–}11)$$

The first form, (9–10), shows that the equations are actually restrictions on the elongations. One can interpret (9–10) as a compatibility condition, i.e., it must be satisfied in order for the bars to *fit* in the *deformed* structure defined by $\mathbf{U}_1$. The second form, (9–11), follows when we express the elongations in terms of the bar forces. Finally, we substitute for $\mathbf{F}_1$ and write the result as

$$\mathbf{f}_{22}\mathbf{F}_2 = \mathbf{d}_2 \qquad (9\text{–}12)$$

where

$$\mathbf{f}_{22} = \mathbf{f}_2 + \mathbf{F}_{1,F_2}^T\mathbf{f}_1\mathbf{F}_{1,F_2} \qquad (9\text{–}13)$$
$$\mathbf{d}_2 = -\mathbf{e}_{2,0} - \mathbf{F}_{1,F_2}^T(\mathbf{e}_{1,0} + \mathbf{f}_1\mathbf{F}_{1,0}) + \mathbf{P}_{2,F_2}^T\bar{\mathbf{U}}_2$$

The coefficient matrix, f_{22}, is called the flexibility matrix for $\mathbf{F}_2$. One can show that $\mathbf{f}_{22}$ is *positive definite* when the bar flexibility factors in $\mathbf{f}_2$ are all *positive*.†

If the material is physically nonlinear, $\mathbf{f}_n$ and $\mathbf{e}_{0,n}$ depend on $\mathbf{F}_n$. Iteration is minimized by applying the loading in increments and approximating the force-elongation relation with a piecewise linear representation. The *incremental* equations are similar in form to the *total* equations.‡ We just have to replace the force, displacement, and elongation terms with their incremental values and interpret **f** as a segmental (tangent) flexibility.

At this point, we summarize the steps involved in the force method.

1. *Determination of* $\mathbf{F}_{1,0}$, $\mathbf{P}_{2,0}$, $\mathbf{F}_{1,F_2}$, *and* $\mathbf{P}_{2,F2}$

We select a stable primary structure $\mathbf{F}_1$ and determine the bar forces and reactions due to $\bar{\mathbf{P}}_1$ and a unit value of each force redundant. This step involves $q + 1$ force analyses on the *primary* structure. Note that we obtain the primary structure by deleting $q = m - n_d$ bars. The selection of a primary structure and solution of the force equilibrium equations can be completely automated.§

† See Prob. 9–1.
‡ See Prob. 9–4.
§ We reduce $\mathscr{B}_1$ to an echelon matrix. See (1–61).

2. *Determination of* $\mathbf{F}_2$, $\mathbf{F}_1$, *and* $\mathbf{P}_2$

We assemble $\mathbf{f}_{22}$, $\mathbf{d}_2$, and solve $\mathbf{f}_{22}\mathbf{F}_2 = \mathbf{d}_2$ for $\mathbf{F}_2$. Then, we determine $\mathbf{F}_1$ and $\mathbf{P}_2$ by combining the $q + 1$ basic solutions.

$$\mathbf{F}_1 = \mathbf{F}_{1,0} + \mathbf{F}_{1,F_2}\mathbf{F}_2$$
$$\mathbf{P}_2 = \mathbf{P}_{2,0} + \mathbf{P}_{2,F_2}\mathbf{F}_2$$

3. *Determination of* $\mathbf{U}_1$

Once $\mathbf{F}_1$ is known, we can evaluate $\mathbf{e}_1$,

$$\mathbf{e}_1 = \mathbf{e}_{1,0} + \mathbf{f}_1\mathbf{F}_1$$

and then solve (9–8),

$$\mathbf{B}_{11}^T\mathbf{U}_1 = \mathbf{e}_1 - \mathbf{B}_{21}^T\bar{\mathbf{U}}_2$$

for $\mathbf{U}_1$.

If only a limited number of displacement components are desired, one can determine these components without actually solving (9–8). To show this, we write $\mathbf{U}_1$ as

$$\mathbf{U}_1 = (\mathbf{B}_{11}^{-1})^T\mathbf{e}_1 - (\mathbf{B}_{21}\mathbf{B}_{11}^{-1})^T\bar{\mathbf{U}}_2 \qquad \text{(a)}$$

We see from (9–15) that the kth column of $\mathbf{B}_{11}^{-1}$ contains the bar forces in the *primary* structure due to a unit value of the kth element in $\mathbf{P}_1$. Also, it follows from (9–6) that the kth column of $\mathbf{B}_{21}\mathbf{B}_{11}^{-1}$ contains the reactions due to a unit value of the kth element in $\mathbf{P}_1$. Now, we obtain the kth element in $\mathbf{U}_1$ (which corresponds to the kth element in $\mathbf{P}_1$) by multiplying the kth column of $\mathbf{B}_{11}^{-1}$ by $\mathbf{e}_1^T$, the kth column of $\mathbf{B}_{21}\mathbf{B}_{11}^{-1}$ by $\bar{\mathbf{U}}_2^T$, and adding the two scalars. Then, letting

$$\begin{aligned} \mathbf{F}_{1,p_{jk}} &= \mathbf{F}_1 \quad \text{due to an unit value of } p_{jk} \text{ with } \mathbf{F}_2 = \mathbf{0} \\ \mathbf{P}_{2,p_{jk}} &= \mathbf{P}_2 \quad \text{due to an unit value of } p_{jk} \text{ with } \mathbf{F}_2 = \mathbf{0} \end{aligned} \qquad (9\text{–}14)$$

we can write the expression for u_{jk} as

$$u_{jk} = \mathbf{F}_{1,p_{jk}}^T\mathbf{e}_1 - \mathbf{P}_{2,p_{jk}}^T\bar{\mathbf{U}}_2 \qquad (9\text{–}15)$$

Note that one works with the statically determinant *primary* structure to determine the displacements.

Example 9–1

Step 1: ***Determination of*** $\mathbf{F}_{1,0}$, $\mathbf{P}_{2,0}$, $\mathbf{F}_{1,F_2}$, ***and*** $\mathbf{P}_{2,F2}$

For the truss shown in Fig. E9–1A,

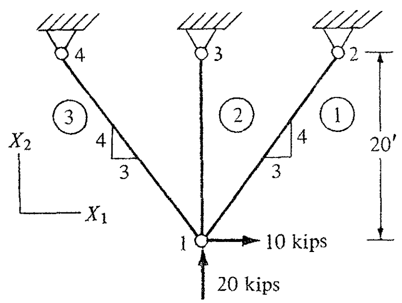
$$n_d = 2 \qquad m = 3 \qquad q = 1$$

We take F_3 as the redundant bar force:

$$\mathbf{F}_1 = \{F_1, F_2\} \qquad \mathbf{F}_2 = \{F_3\}$$

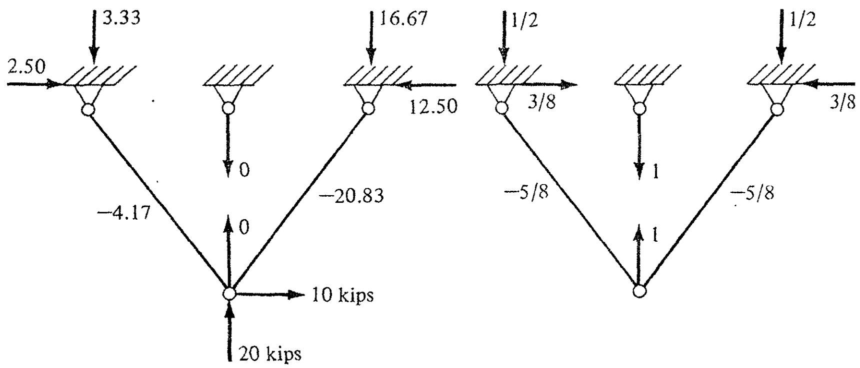
The primary structure consists of bars 1 and 2. Note that all force analyses are performed on the *primary* structure. The forces and reactions corresponding to $\mathbf{P}_1$ and $F_3 = +1$ can be readily obtained using the method of joints. The results are shown in Fig. E9–1B.

Fig. E9–1A



(1) $A_1 = 1.0 \text{ in.}^2 \quad A_2 = 0.5 \text{ in.}^2 \quad A_3 = 0.5 \text{ in.}^2$
(2) Material is linearly elastic. $E = 3 \times 10^4$ ksi for all bars.
(3) $e_{0,1} = -1/16$ in. $e_{0,2} = e_{0,3} = 0$.
(4) $u_{32} + 1/10$ in. $u_{41} = -1/15$ in.

Fig. E9–1B



We could have obtained the above results for $\mathbf{F}_1$ by solving

$$\mathbf{B}_{11}\mathbf{F}_1 = \bar{\mathbf{P}}_1 - \mathbf{B}_{12}\mathbf{F}_2$$

which, for this system, has the form

$$\begin{bmatrix} -.6 & +.6 \\ -.8 & +.8 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 10 \\ 20 \end{Bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \{F_3\}$$

Step 2: Determination of $\mathbf{f}_{22}$, $\mathbf{d}_2$, $\mathbf{F}_1$, *and* $\mathbf{F}_2$

Since only u_{32} and u_{41} are finite, we can contract $\bar{\mathbf{U}}_2$ and $\mathbf{P}_2$,

$$\mathbf{P}_2' = \{p_{32}, p_{41}\}$$
$$\bar{\mathbf{U}}_2' = \{u_{32}, u_{41}\}$$

and write

$$\mathbf{P}_{2,F_2}^T\bar{\mathbf{U}}_2 = (\mathbf{P}'_{2,F_2})^T\bar{\mathbf{U}}'_2$$

The force matrices follow from step 1:

$$\mathbf{F}_{1,0} = \{-20.83, -4.17\} \quad \text{(kips)}$$
$$\mathbf{F}_{1,F_2} = \{-\tfrac{5}{8}, -\tfrac{5}{8}\} \quad \text{(kips)}$$
$$\mathbf{P}'_{2,F_2} = \{+1, +\tfrac{3}{8}\} \quad \text{(kips)}$$

Also, we are given that

$$\mathbf{e}_{1,0} = \{e_{0,1}, e_{0,2}\} = \{-\tfrac{1}{16}, 0\} \quad \text{(inches)}$$
$$\mathbf{e}_{2,0} = \{e_{0,3}\} = 0$$
$$\bar{\mathbf{U}}_2 = \{+\tfrac{1}{10}, -\tfrac{1}{15}\} \quad \text{(inches)}$$

It remains to assemble $\mathbf{f}_1$, $\mathbf{f}_2$ and evaluate $\mathbf{f}_{22}$ and $\mathbf{d}_2$.
The flexibility factors are (in./kip)

$$f_1 = \frac{12(25)}{3 \times 10^4} \qquad f_2 = \frac{12(25)}{1.5 \times 10^4} \qquad f_3 = \frac{12(20)}{1.5 \times 10^4}$$

Then,

$$\mathbf{f}_1 = \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}(2 \times 10^{-2})$$
$$\mathbf{f}_2 = [f_3] = 0.8(2 \times 10^{-2})$$

Evaluating the various products in (9–13), (9–12) reduces to

$$1.38F_3 = -7.31 \tag{a}$$

Solving (a), we obtain

$$\mathbf{F}_2 = \{\mathbf{F}_3\} = -5.27 \text{ kips}$$
$$\mathbf{F}_1 = \mathbf{F}_{1,0} + \mathbf{F}_{1,F_2}\mathbf{F}_2 = \begin{Bmatrix} -17.53 \text{ kips} \\ -\ 0.87 \text{ kips} \end{Bmatrix}$$

Equation (a) actually represents a restriction on the elongations. The original form of (a) follows from (9–10).

$$e_3 - \tfrac{5}{8}e_1 - \tfrac{5}{8}e_2 = +\tfrac{3}{40} \tag{b}$$

Equation (b) reduces to (a) when we substitute for the elongations in terms of the bar forces.

Step 3: Determination of the Displacements

Suppose only u_{11} is desired. Using (9–15),

$$u_{11} = \mathbf{F}_{1,p_{11}}^T\mathbf{e}_1 - (\mathbf{P}'_{2,p_{11}})^T\bar{\mathbf{U}}'_2 \tag{c}$$

Now,

$$\bar{\mathbf{U}}'_2 = \{\tfrac{1}{10}, -\tfrac{1}{15}\}$$
$$\mathbf{e}_1 = \mathbf{e}_{1,0} + \mathbf{f}_1\mathbf{F}_1 = \{-.24, -.018\}$$

We apply a unit load at joint 1 in the X_1 direction and determine the bar forces in the *primary* structure and the reactions (p_{32}, p_{41}) corresponding to the nonvanishing prescribed displacements:

$$\mathbf{F}_{1,p_{11}} = \{-\tfrac{5}{6}, \tfrac{5}{6}\}$$
$$\mathbf{P}_{2,p_{11}} = \{0, -\tfrac{1}{2}\}$$

Substituting in (c), we obtain

$$u_{11} = +.185 - .033 = +.15 \text{ in}$$

If both displacement components are desired, we apply (9–15) twice. This is equivalent to solving (9–8).

9–3. GOVERNING EQUATIONS—VARIATIONAL APPROACH

We obtained the elongation compatibility equations (9–10) by operating on the elongation-displacement equations. Alternatively, one can use the principle of virtual forces developed in Sec. 7–3. It is shown there (see Equation (7–14)) that the true elongations satisfy the condition,

$$\Delta \mathbf{F}^T \mathbf{e} - \Delta \mathbf{P}_2^T \bar{\mathbf{U}}_2 = 0 \tag{a}$$

for any statically permissible system of virtual bar forces and reactions which satisfy the constraint condition,

$$\mathbf{B}_1 \, \Delta \mathbf{F} = \Delta \mathbf{P}_1 = \mathbf{0} \tag{b}$$

Equation (b) states that the virtual bar forces cannot lead to increments in the *prescribed* joint loads, i.e., they must be self-equilibrating.

Now, using (9–4), (9–5), we can write

$$\mathbf{F} = \begin{Bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_{1,0} \\ \mathbf{0} \end{Bmatrix} + \begin{bmatrix} \mathbf{F}_{1,F_2} \\ \mathbf{I}_q \end{bmatrix} \mathbf{F}_2 \tag{c}$$

where

$$\mathbf{B}_1 \begin{Bmatrix} \mathbf{F}_{1,0} \\ \mathbf{0} \end{Bmatrix} = \bar{\mathbf{P}}_1 \tag{d}$$

and

$$\mathbf{B}_1 \begin{Bmatrix} \mathbf{F}_{1,F_2} \\ \mathbf{I}_q \end{Bmatrix} = \mathbf{0} \tag{e}$$

Then

$$\Delta \mathbf{F} = \begin{Bmatrix} \mathbf{F}_{1,F_2} \\ \mathbf{I}_q \end{Bmatrix} \Delta \mathbf{F}_2 \tag{f}$$

satisfies (b) for arbitrary $\Delta \mathbf{F}_2$. The reactions due to $\Delta \mathbf{F}_2$ are obtained from (9–6):

$$\Delta \mathbf{P}_2 = \mathbf{B}_2 \, \Delta \mathbf{F} = \mathbf{P}_{2,F_2} \, \Delta \mathbf{F}_2 \tag{g}$$

Substituting for $\Delta \mathbf{F}$ and $\Delta \mathbf{P}_2$, (a) expands to

$$\Delta \mathbf{F}_2^T (\mathbf{F}_{1,F_2}^T \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{P}_{2,F_2}^T \bar{\mathbf{U}}_2) = 0 \tag{h}$$

Equation (h) must be satisfied for arbitrary $\Delta \mathbf{F}_2$. Finally, it follows that

$$\mathbf{F}_{1,F_2}^T \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{P}_{2,F2}^T \bar{\mathbf{U}}_2 = \mathbf{0} \tag{i}$$

Equation (i) is identical to (9–10). Note that the elongation compatibility

equations are *independent* of the material behavior. If the material is physically linear, (i) leads to a set of q linear equations in $\mathbf{F}_2$ when we substitute for the elongations in terms of the bar forces.

We determine the displacements by applying the general form of the principle of virtual forces (see (7–10))

$$\Delta \mathbf{F}^T \mathbf{e} - \Delta \mathbf{P}_2^T \bar{\mathbf{U}}_2 = \Delta \mathbf{P}_1^T \bar{\mathbf{U}}_1 \tag{j}$$

where the virtual forces satisfy the force-equilibrium equations,

$$\begin{aligned} \Delta \mathbf{P}_1 &= \mathbf{B}_1 \, \Delta \mathbf{F} \\ \Delta \mathbf{P}_2 &= \mathbf{B}_2 \, \Delta \mathbf{F} \end{aligned} \tag{k}$$

Since only $\mathbf{F}_1$ is required to equilibrate $\mathbf{P}_1$, we can take

$$\begin{aligned} \Delta \mathbf{F}_1 &= \mathbf{F}_{1,P_1} \, \Delta \mathbf{P}_1 \\ \Delta \mathbf{F}_2 &= \mathbf{0} \end{aligned} \tag{l}$$

and (j) leads to

$$\mathbf{U}_1 = \mathbf{F}_{1,P_1}^T \mathbf{e}_1 - \mathbf{P}_{2,P_1}^T \bar{\mathbf{U}}_2 \tag{m}$$

Note that

$$\mathbf{F}_{1,P_1} = \mathbf{B}_{11}^{-1} \tag{n}$$

One can interpret the compatibility equations expressed in terms of $\mathbf{F}_2$ as the Euler equations for the total complementary energy function,

$$\Pi_c = V_T^* - \mathbf{P}_2^T \bar{\mathbf{U}}_2 = \Pi_c(\mathbf{F}_2) \tag{o}$$

This approach is discussed in sec. 7–5. We take $\mathbf{X} = \mathbf{F}_2$ in (7–35). Then,

$$\mathbf{F}_{,x} = \begin{bmatrix} \mathbf{F}_{1,F_2} \\ \mathbf{I}_q \end{bmatrix} \qquad \mathbf{P}_{2,x} = \mathbf{P}_{2,F_2} \tag{p}$$

and (7–37) coincides with (i). We have written the expanded form of (i) as

$$\mathbf{f}_{22} \mathbf{F}_2 = \mathbf{d}_2 \tag{q}$$

Since (i) are the Euler equations for Π_c,

$$d\Pi_c = \Delta \mathbf{F}_2^T (\mathbf{f}_{22} \mathbf{F}_2 - \mathbf{d}_2) \tag{r}$$

and it follows that

$$\Pi_c = \tfrac{1}{2} \mathbf{F}_2^T \mathbf{f}_{22} \mathbf{F}_2 - \mathbf{F}_2^T \mathbf{d}_2 \tag{s}$$

for the linearly elastic case. One can show that the stationary point corresponds to a relative minimum value of Π_c when the tangent flexibility factors for the redundant bars are all positive.†

9–4. COMPARISON OF THE FORCE AND MESH METHODS

It is of interest to compare the force method for a truss with the procedure followed to find the currents in an electrical network. The latter involves the

† See Prob. 9–8.

application of Kirchhoff's laws and is called the mesh method. Various phases of the electrical network formulation are discussed in Probs. 6–6, 6–14, and the governing equations for a linear resistance d-c network are developed in Probs. 6–14, 6–23. We list the notation and governing equations for convenience:

b = number of branches
n = number of nodes
$N = n - 1$
$M = b - N = b - n + 1$
V_j = potential at node j with respect to the reference potential, v_n.
k_+, k_- = nodes at *positive* and *negative* ends of branch k
i_k = current in branch k, positive when directed *from* node k_- *to* node k_+
e_k = potential drop for branch $k = V_{k_-} - V_{k_+}$
$e_{0,k}$ = emf for branch k
R_k = resistance for branch k

The governing equations expressed in matrix notation are (see Prob. 6–23):

$$\mathbf{A}^T\mathbf{i} = \mathbf{0} \qquad (N \text{ eqs.}) \qquad (9\text{–}16)$$
$$\mathbf{e} = \mathbf{AV} = \mathbf{e}_0 + \mathbf{Ri} \qquad (b \text{ eqs.}) \qquad (9\text{–}17)$$

where

$$\begin{aligned} \mathbf{i} &= \{i_1, i_2, \ldots, i_b\} \\ \mathbf{e} &= \{e_1, e_2, \ldots, e_b\} \\ \mathbf{V} &= \{V_1, V_2, \ldots, V_N\} \\ \mathbf{R} &= \begin{bmatrix} R_1 & & \\ & R_2 & \\ & & \ddots \\ & & & R_b \end{bmatrix} \end{aligned} \qquad (9\text{–}18)$$

and $\mathbf{A}$ is obtained by deleting the *last* column of the branch-node connectivity matrix $\mathscr{A}$. Note that $\mathscr{A}$ has only two entries in any row. For row k ($k = 1, 2, \ldots, b$),

$$\begin{aligned} \mathscr{A}kk_- &= +1 \\ \mathscr{A}kk_+ &= -1 \qquad j \neq k_+ \text{ or } k_- \\ \mathscr{A}_{kj} &= 0 \qquad j = 1, 2, \ldots, N \end{aligned} \qquad (9\text{–}19)$$

Actually, $\mathscr{A}$ is just the matrix equivalent of the branch-node connectivity table.

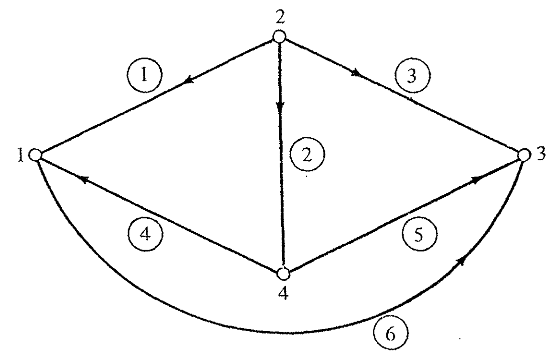
Example 9–2

A network can be represented by a line drawing consisting of curves interconnected at various points. The curves and intersection points are conventionally called branches and nodes respectively. Each branch is terminated at *two* different nodes and no two branches have a point in common which is *not* a node. Also, two nodes are connected by at least one path. A collection of nodes and branches satisfying the above restrictions is called a *linear connected graph.* If each branch is assigned a direction, the graph is said to be

oriented. The connectivity relations for a network are topological properties of the corresponding oriented graph.

Consider the oriented graph shown. We list the branch numbers vertically and the node numbers horizontally. We assemble $\mathscr{A}$ working with successive branches. Finally, we obtain $\mathbf{A}$ by deleting the last column (col. 4) of $\mathscr{A}$.

Fig. E9–2



Branch	Node 1	Node 2	Node 3
1	−1	+1	
2		+1	
3		+1	−1
4	−1		
5			−1
6	+1		−1

$\mathbf{A}$

b

N

Now, $\mathbf{A}$ has N linearly independent *columns.* Therefore, it is possible to solve (9–16) for N branch currents in terms of $b - N = M$ branch currents. We suppose the branches are numbered such that the first N rows of $\mathbf{A}$ contain a nonvanishing determinant of order N and partition $\mathbf{A}$, $\mathbf{i}$ after row N.

$$\underset{(b\times N)}{\mathbf{A}} = \begin{bmatrix} \overset{(N\times N)}{\mathbf{A}_1} \\ \hline \underset{(M\times N)}{\mathbf{A}_2} \end{bmatrix} \qquad \underset{(b\times 1)}{\mathbf{i}} = \begin{Bmatrix} \overset{(N\times 1)}{\mathbf{i}_1} \\ \hline \underset{(M\times 1)}{\mathbf{i}_2} \end{Bmatrix} \qquad (9\text{–}20)$$

Introducing (9–20) in (9–16) leads to

$$\mathbf{A}_1^T\mathbf{i}_1 = -\mathbf{A}_2^T\mathbf{i}_2 \qquad (9\text{–}21)$$

Since $|\mathbf{A}_1| \neq 0$, we can solve for $\mathbf{i}_1$ in terms of $\mathbf{i}_2$. We write the solution of the node equations as

$$\begin{gathered}\mathbf{i} = \mathbf{C}\mathbf{i}_2 \\ \Downarrow \\ \begin{Bmatrix}\mathbf{i}_1 \\ \hline \mathbf{i}_2\end{Bmatrix} = \begin{bmatrix}\mathbf{C}_1 \\ \hline \mathbf{I}_M\end{bmatrix}\mathbf{i}_2\end{gathered} \qquad (9\text{–}22)$$

Note that $\mathbf{C}_1$ is of order N by M and is related to $\mathbf{A}_1$, $\mathbf{A}_2$ by

$$\mathbf{C}_1 = -(\mathbf{A}_2\mathbf{A}_1^{-1})^T \qquad (9\text{–}23)$$

It remains to determine a set of M equations for $\mathbf{i}_2$.

One can express (9–17) in partitioned form and then eliminate $\mathbf{V}$, or alternatively, one can use the variational principle developed in Prob. 7–6. Using the first approach, we write (9–17) as

$$\begin{aligned}\mathbf{e}_1 &= \mathbf{A}_1\mathbf{V} = \mathbf{e}_{1,0} + \mathbf{R}_1\mathbf{i}_1 \qquad (N \text{ eqs}) \\ \mathbf{e}_2 &= \mathbf{A}_2\mathbf{V} = \mathbf{e}_{2,0} + \mathbf{R}_2\mathbf{i}_2 \qquad (M \text{ eqs})\end{aligned} \qquad (9\text{–}24)$$

Once $\mathbf{i}_1$ is known, we can find $\mathbf{V}$ from

$$\mathbf{A}_1\mathbf{V} = \mathbf{e}_1 = \mathbf{e}_{1,0} + \mathbf{R}_1\mathbf{i}_1 \qquad (9\text{–}25)$$

Eliminating $\mathbf{V}$ from the second equation in (9–24) and using (9–23), we obtain

$$\mathbf{e}_2 + \mathbf{C}_1^T\mathbf{e}_1 = \mathbf{0} \qquad (9\text{–}26)$$

Equation (9–26) represents M equations relating the branch potential differences (voltages). Finally, substituting for $\mathbf{e}_j$ in terms of $\mathbf{i}_j$ leads to

$$(\mathbf{R}_2 + \mathbf{C}_1^T\mathbf{R}_1\mathbf{C}_1)\mathbf{i}_2 = -\mathbf{e}_{2,0} - \mathbf{C}_1^T\mathbf{e}_{1,0} \qquad (9\text{–}27)$$

The coefficient matrix for $\mathbf{i}_2$ is positive definite when the branch resistances are positive. This will be the case for a *real* system.

The essential step in the solution involves solving (9–21), that is, finding $\mathbf{C}_1$. Note that $\mathbf{C}_1$ corresponds to $\mathbf{F}_{1,F_2}$ for the truss problem. Also, the branches comprising $\mathbf{A}_1$ (and $\mathbf{i}_1$) correspond to the *primary* structure. Although the equations for the truss and electrical network are similar in form, it should be noted that the network problem is one dimensional whereas the truss problem involves the *geometry* as well as the connectivity of the system. One can assemble $\mathbf{C}_1$ using only the topological properties of the oriented graph which represents the network. To find the corresponding matrices ($\mathbf{F}_{1,0}$ and $\mathbf{F}_{1,F_2}$) for a truss, one must solve a system of linear equations. In what follows, we describe a procedure for assembling $\mathbf{C}_1$ directly from the oriented graph.

A closed path containing only *one* repeated node that begins and ends at that node is called a *mesh*. One can represent a mesh by listing sequentially the branches traversed. A *tree* is defined as a connected graph having *no* meshes. Let b_T be the number of branches in a tree connecting n nodes. One can easily show that

$$b_T = n - 1 = N \qquad (9\text{–}28)$$

We reduce a graph to a tree by removing a sufficient number of branches such that *no* meshes remain. The branches removed are generally called *chords*. The required number of chords is equal to $b - b_T = b - N = M$. Now, we associate the branches comprising a tree with the rows of $\mathbf{A}_1$. Selecting a tree is equivalent to selecting N linearly independent rows in $\mathbf{A}$. The M chords correspond to the redundant branches, that is, the rows of $\mathbf{A}_2$. Note that one can always number the branches such that the first N branches define a tree.

Chord j and the unique path (in the tree) connecting the terminals of chord j define a mesh, say mesh j. We take the positive direction of mesh j (clockwise or counterclockwise) such that the mesh direction coincides with the positive direction for chord j. Now, the current is *constant* in a mesh. Suppose branch r is contained in mesh j. Then, the current in branch r due to a unit value of i_j is equal to $+1$ (-1) if the positive directions of branch r and mesh j coincide (are opposite in sense).

We have expressed the solution of the node equations as

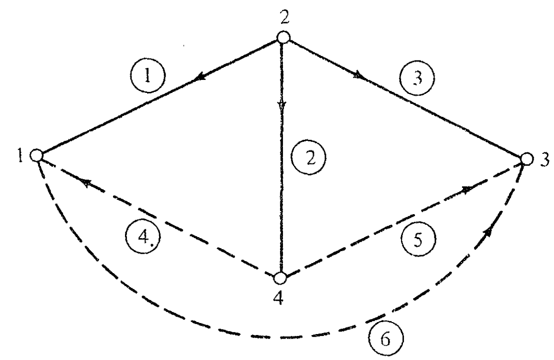
$$\underset{(N\times 1)}{\mathbf{i}_1} = \underset{(N\times M)}{\mathbf{C}_1}\;\underset{(M\times 1)}{\mathbf{i}_2}$$

Now, we take the elements of $\mathbf{i}_2$ as the *chord* (mesh) currents. Then $\mathbf{i}_1$ represents the required branch currents in the *tree*. We assemble $\mathbf{C}_1$ working with the columns. The column corresponding to $\mathbf{i}_j$ involves only those branches of the *tree* which are contained in mesh j. We enter $(+1, -1, 0)$ in row k of this column if branch k is (positively, negatively, not) included in mesh j.

Example 9–3

For the graph in example 9–2, $N = n - 1 = 3$ and $b = 6$. Then $M = b - N = 3$ and we must remove 3 branches to obtain a tree. We take branches 4, 5, and 6 as the chords. The resulting tree is shown in Fig. E9–3. We indicate the chords by dashed lines.

Fig. E9–3



For this selection of a tree,

$$\mathbf{i}_1 = \{i_1, i_2, i_3\} \qquad \mathbf{i}_2 = \{i_4, i_5, i_6\}$$

The meshes associated with the chords follow directly from the sketch:

$$\begin{array}{ll} \text{mesh 4} & ④, -①, +② \\ \text{mesh 5} & ⑤, -③, +② \\ \text{mesh 6} & ⑥, -③, +① \end{array} \tag{a}$$

To assemble $\mathbf{C}_1$ we list the branches of the tree vertically and the chord numbers horizontally. We work with successive columns, that is, successive chords. The resulting matrix is listed below. Note that $\mathbf{C}_1$ is just the matrix equivalent of (a).

		← Chords →		
		4	5	6
↑ Branches of the tree ↓	1	−1	0	+1
	2	+1	+1	0
	3	0	−1	−1

The matrices, $\mathbf{A}_1$ and $\mathbf{A}_2$, follow from Example 9–2:

$$\mathbf{A}_1 = \begin{bmatrix} -1 & +1 & 0 \\ 0 & +1 & 0 \\ 0 & +1 & -1 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ +1 & 0 & -1 \end{bmatrix}$$

One can readily verify that

$$\mathbf{C}_1 = -(\mathbf{A}_2\mathbf{A}_1^{-1})^T$$

The matrix, $\mathbf{C} = \{\mathbf{C}_1, \mathbf{I}_m\}$, is called the branch-mesh incidence matrix. Using (9–23), we see that $\mathbf{A}$ and $\mathbf{C}$ have the property

$$\mathbf{A}^T\mathbf{C} = \overset{(N \times M)}{\mathbf{0}} \tag{9–29}$$

Also, we can express the compatibility equations, (9–26), as

$$\mathbf{C}^T\mathbf{e} = \overset{(M \times 1)}{\mathbf{0}} \tag{9–30}$$

The rows of $\mathbf{C}^T$ define the incidence of the meshes on the branches. Equation (9–30) states that the sum of the potential drops around each mesh must be zero and is just Kirchhoff's voltage law expressed in matrix form. The matrix formulation of the network problem leads to the *same* system of equations that one would obtain by applying Kirchhoff's current and voltage laws to the various nodes and meshes. This, of course, also applies to the truss problem. The two approaches differ only with respect to the assemblage of the governing equations. In the conventional approach, one assembles the equations individually. This involves repeated application of the basic laws. When the equations are expressed in matrix form, the steps reduce to a sequence of matrix multiplications.

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PROBLEMS

9–1. Show that the coefficient matrix $\mathbf{f}_{22}$ is positive definite for arbitrary rank of $\mathbf{F}_{1,F_2}$ when $\mathbf{f}_2$ is positive definite. Use the approach suggested in Problems 2–12 through 2–14.

9–2. Solve the following system using the procedure outlined in Sec. 9–2. Take $\mathbf{X}_1 = \{x_1, x_2\}$

$$\begin{bmatrix} 2 & 2 & 3 & 1 \\ 1 & 2 & 4 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 3 \end{Bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 2 & 2 \\ 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{2}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} + \begin{Bmatrix} 3 \\ 3 \\ 1 \\ 2 \end{Bmatrix}$$

9–3. Consider a system of m equations in n unknowns, $\mathbf{ax} = \mathbf{c}$, where $m > n$. Suppose $r(\mathbf{a}) = n$ and the first n rows of $\mathbf{a}$ are linearly independent. Let $q = m - n$.

(a) Show that the consistency requirement for the system leads to q relations between the elements of $\mathbf{c}$.

(b) Interpret (9–10) from this point of view.

9–4. Develop an incremental "force" formulation starting with

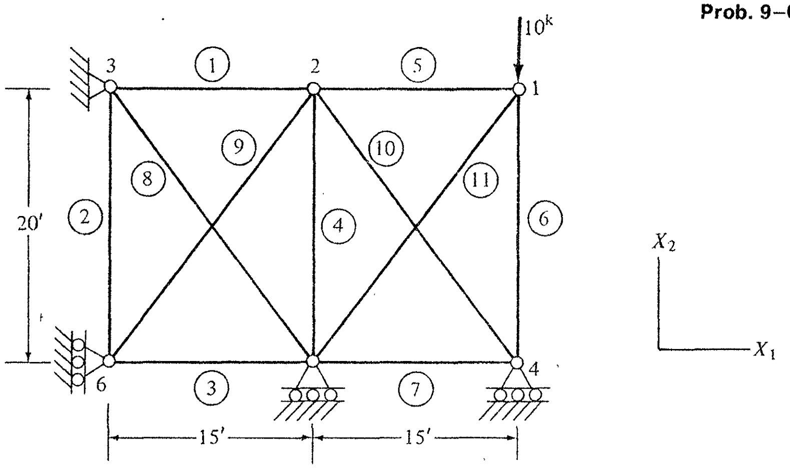
$$\Delta \bar{\mathbf{P}}_1 = \mathbf{B}_1 \, \Delta \mathbf{F}$$
$$\Delta \mathbf{P}_2 = \mathbf{B}_2 \, \Delta \mathbf{F}$$
$$\Delta \mathbf{e} = \mathbf{B}_1^T \, \Delta \mathbf{U}_1 + \mathbf{B}_2^T \, \Delta \bar{\mathbf{U}}_2 = \Delta \mathbf{e}_0^t + \mathbf{f}^t \, \Delta \mathbf{F}$$

where f^t, Δe_0^t represent the flexibility factor and incremental initial elongation for the segment corresponding to the initial value of F. One has to modify both f^t and Δe_0^t if the limit of the segment is exceeded (see sec. 6–4 for a detailed treatment).

Consider the case where the loading distribution is constant, i.e., where only the magnitude is increased. Let $\mathbf{P}_1 = \lambda \psi$ where λ is the load parameter and ψ defines the loading distribution. Discuss how you would organize the computational scheme. Also discuss how you would account for either yielding or buckling of a bar. Distinguish between a redundant bar and a bar in the primary structure.

9–5. Solve Prob. 8–3 with the force method. Take F_3 as the force redundant.

9–6. Assemble the equations for $\mathbf{F}_2 = \{F_8, F_9, F_{10}, F_{11}\}$ for the truss shown.



Prob. 9–6

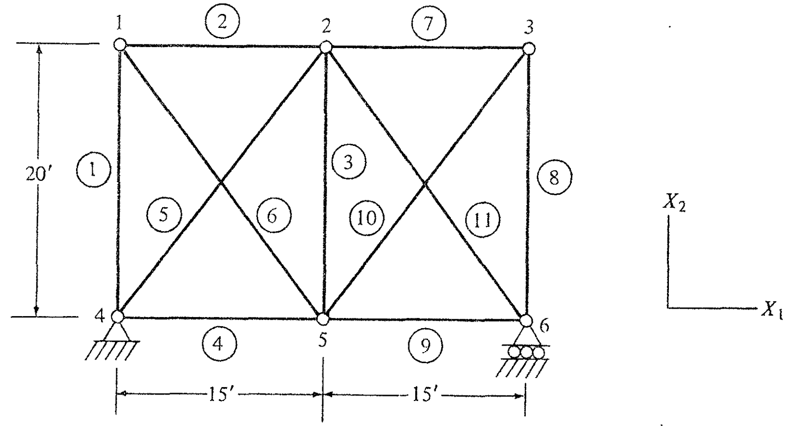
(1) Material is linear elastic and the flexibility factors are *equal*.
(2) Only u_{42} is finite. Take $\bar{\mathbf{U}}_2' = \{u_{42}\} = u_{42}$.
(3) Only initial elongation for bar 4.

9–7. For the truss shown:
(a) Using (9–10), determine the elongation-compatibility relations. Take bars ⑥, ⑩ as the redundant bars.
(b) Express u_{52} in terms of the elongations and support movements.

9–8. By definition (see (7–26) and (7–31))

$$d\Pi_c = \Delta \mathbf{F}^T \mathbf{e} - \Delta \mathbf{P}_2^T \bar{\mathbf{U}}_2$$

Prob. 9–7

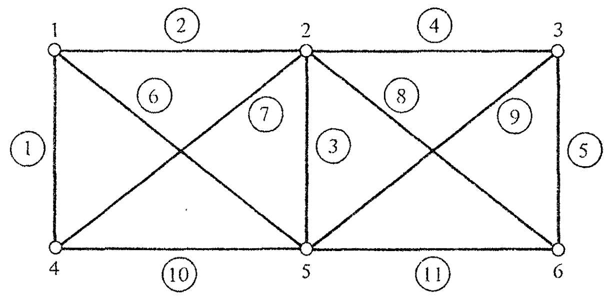


Then

$$d^2\Pi_c = d(d\Pi_c) = \Delta \mathbf{F}^T \, \mathbf{de}$$

Express $d^2\Pi_c$ as a quadratic form in $\Delta \mathbf{F}_2$. Consider the material to be nonlinear elastic and establish criteria for the stationary point to be a relative minimum.

9–9. Consider the oriented linear graph shown.



Prob. 9–9

(a) Determine **A**.
(b) Determine **C**.
(c) Verify that $\mathbf{A}^T\mathbf{C} = \mathbf{0}$.