12.005 Lecture Notes 17

Special Cases

In general $\tau_{ij} = \lambda e_{ii}\delta_{ij} + 2\mu e_{ij}$

Plane stress: (e.g., $\tau_{zz} = 0, \quad \tau_{xz} = 0$)

$$\underset{\sim}{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \tau_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$p = -\frac{\tau_{xx} + \tau_{yy}}{3}$$

$$\underset{\sim}{\tau}^{\text{dev}} = \begin{bmatrix} \frac{2\tau_{xx} - \tau_{yy}}{3} & \tau_{xy} & 0 \\ \tau_{xy} & \frac{2\tau_{yy} - \tau_{xx}}{3} & 0 \\ 0 & 0 & \frac{-\tau_{xx} - \tau_{yy}}{3} \end{bmatrix}$$

$$\tau_{zz}^{\text{dev}} = -\frac{\tau_{xx} + \tau_{yy}}{3}$$

Solving for strains:

$$e_{xx} = \frac{1}{E}(\tau_{xx} - \nu\tau_{yy})$$
$$e_{yy} = \frac{1}{E}(\tau_{yy} - \nu\tau_{xx})$$
$$e_{zz} = -\frac{\nu}{E}(\tau_{xx} + \tau_{yy})$$
$$e_{xy} = \frac{\tau_{xy}}{2\mu}$$

Plane strain: (e.g., $e_{zz} = e_{xz} = 0 = \tau_{zz}{}^{\text{dev}} \quad \Rightarrow \quad \tau_{xx}{}^{\text{dev}} = -\tau_{yy}{}^{\text{dev}}$)

$$\underset{\sim}{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \tau_{yy} & 0 \\ 0 & 0 & \dfrac{\tau_{xx} + \tau_{yy}}{2} \end{bmatrix}$$

$$p = -\tau_{zz}$$

$$\underset{\sim}{\tau}^{\text{dev}} = \begin{bmatrix} \dfrac{\tau_{xx} - \tau_{yy}}{2} & \tau_{xy} & 0 \\ \tau_{xy} & \dfrac{\tau_{yy} - \tau_{xx}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e_{xx} = \frac{1+\nu}{E}\left[\tau_{xx}(1-\nu) - \nu\tau_{yy}\right]$$
$$e_{yy} = \frac{1+\nu}{E}\left[\tau_{yy}(1-\nu) - \nu\tau_{xx}\right]$$
$$e_{xy} = \frac{\tau_{xy}}{2\mu}$$

Note:

Plate stress	$\rightarrow$	Plane strain
E	$\rightarrow$	$\dfrac{E}{1-\nu^2}$
ν	$\rightarrow$	$\dfrac{\nu}{1-\nu}$
$\dfrac{2\lambda\mu}{\lambda+2\mu}$	$\leftarrow$	λ

Plane stress: (e.g., $\tau_{33} = 0$)

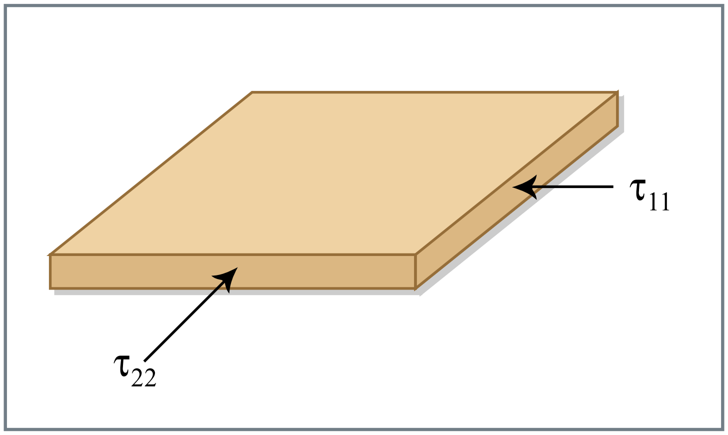

Figure 17.1

Figure by MIT OCW.

$$e_{11} = \frac{1}{E}(\tau_{11} - \nu\tau_{22})$$
$$e_{22} = \frac{1}{E}(\tau_{22} - \nu\tau_{11})$$
$$e_{33} = -\frac{\nu}{E}(\tau_{11} + \tau_{22})$$

Tectonic Stress – Often plane stress is useful (3D$\rightarrow$2D).

Assume mostly in plate stress

Assume

$$\tau_{zz} = -\rho g z \quad \text{(lithostatic)}$$
$$\tau_{xx} = -\rho g z + \Delta\tau_{xx}$$
$$\tau_{yy} = -\rho g z + \Delta\tau_{yy}$$

The non-lithostatic (as opposed to deviatoric) stress:

$$\left.\begin{aligned} \tau_{zz}' &= 0 \\ \tau_{xx}' &= \Delta\tau_{xx} \\ \tau_{yy}' &= \Delta\tau_{yy} \end{aligned}\right\} \quad \text{plane stress}$$

Assume for simplicity $\Delta\tau_{xx} = \Delta\tau_{yy} = \sigma$

$$e_{11} = e_{22} = \frac{(1-\nu)}{E}\sigma \quad \Rightarrow \quad \text{compression for } \sigma \text{ compression}$$

$$e_{33} = -\frac{2\nu}{E}\sigma \quad \Rightarrow \quad \text{extension for } \sigma \text{ compression}$$

$$\text{dilation } \theta = (2-4\nu)\frac{\sigma}{E} = -\frac{\delta\rho}{\rho}$$

Look at the change in lithostatic pressure at a depth $h \rightarrow h + \delta h = h(1 - 2\nu\frac{\sigma}{E})$

$$\rho g h \rightarrow (\rho + \delta\rho) g (h + \delta h)$$

Plane stress, $\sigma_1 = \sigma_2 = \sigma$

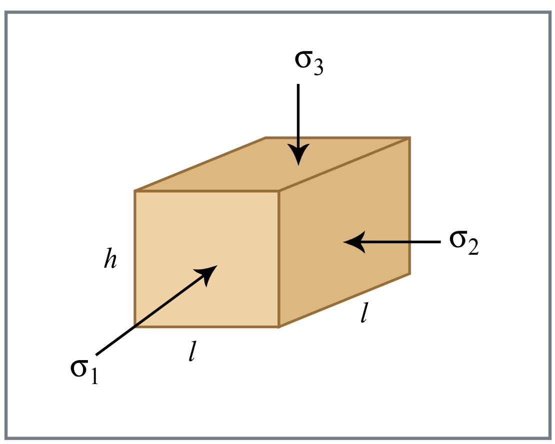

Figure 17.2

Figure by MIT OCW.

$$\varepsilon_1 = \varepsilon_2 = \frac{(1-\nu)}{E}\sigma$$

$$\varepsilon_3 = \frac{-2\nu}{E}\sigma$$

$$\theta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \frac{(2-4\nu)}{E}\sigma = \frac{2(1-2\nu)}{E}\sigma$$

Now to conserve mass, $\frac{\delta\rho}{\rho} = -\frac{\delta V}{V} = \theta$

New lithostat:

$$\begin{aligned}\delta\tau_{zz} &= -(\rho+\delta\rho)g(h+\delta h) + \rho g h \\ &= (1+\theta)g(1+\frac{2\nu}{E}\sigma)\rho h - \rho g h \\ &= (\theta + \frac{2\nu}{E}\sigma)\rho g h\end{aligned}$$

$$\frac{\delta\tau_{zz}}{\sigma} = \frac{2(1-\nu)}{E}\rho g h$$

Use

$$\begin{aligned}&\nu = 0.25 \\ &E = 100 \text{ GPa} = 1 \text{ Mbar} \\ &h = 100 \text{ km} \\ &\rho = 3 \text{ Mg/m}^3 = 3 \text{ g/cm}^3\end{aligned}$$

$$\frac{\delta\tau_{zz}}{\sigma} = \frac{2(1-\nu)}{E}\rho g h \simeq 5\%$$