12.005 Lecture Notes 28

Time Dependent Porous Flow

Assume 1-D flow through an unconfined aquifer.

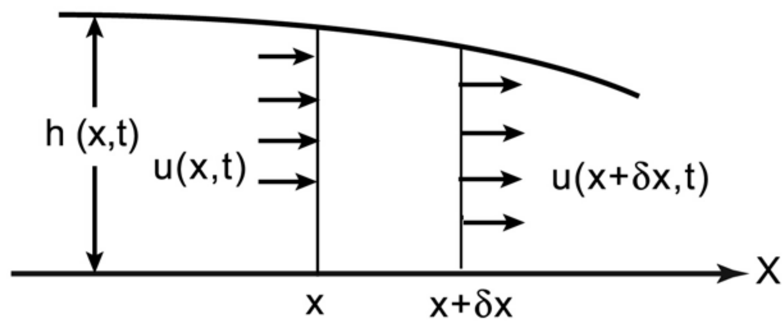

Figure 28.1

Flux in $Q_{in} = u(x,t)h(x,t)$

Flux out $Q_{out} = u(x+\delta x,t)h(x+\delta x,t)$

$$Q_{out} - Q_{in} \approx \frac{\partial}{\partial x}(uh)\delta x \Rightarrow \frac{\partial h}{\partial t}$$

But medium is porous, with porosity ϕ. For a given ΔQ, the smaller ϕ, the larger $\partial h / \partial t$

$$Q_{out} - Q_{in} = -\phi[h(t+\delta t,x) - h(t,x)]\delta t \approx \phi \frac{\partial h}{\partial t}\delta x \delta t$$

Mass conservation $\Rightarrow \phi \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0$

Darcy's law (Dupuit approx) $\Rightarrow u = -\frac{k\rho g}{\eta}\frac{\partial h}{\partial x}$

Combining $\Rightarrow \frac{\partial h}{\partial t} = \frac{k\rho g}{\eta\phi}\frac{\partial}{\partial x}(h\frac{\partial h}{\partial x})$ ("Boussinesq Equation")

This is a nonlinear diffusion equation. We can simplify it by assuming

$h = h_0 + h'$ $\quad h' = h_0$

$$\frac{\partial}{\partial x}(h\frac{\partial h}{\partial x}) = \frac{\partial}{\partial x}[(h_0 + h')\frac{\partial}{\partial x}(h_0 + h')]$$

$$= \frac{\partial}{\partial x}[(h_0 + h')\frac{\partial h'}{\partial x}]$$

$$\approx h_0 \frac{\partial^2 h'}{\partial x^2}$$

This gives the "familiar" diffusion equation

$$\frac{\partial h'}{\partial t} = \frac{k\rho g h_0}{\eta\phi}\frac{\partial^2 h'}{\partial x^2}$$

$\frac{k\rho g h_0}{\eta\phi}$ equivalent to K in heat flow equation

equivalent to $\upsilon = \eta / \rho$ in fluid ½ space

[note inverse relationship of fluid viscosity η!]

Consider the sudden lowering of water in a river channel next to a saturated bank.

$h(x^+,0) = h_0$ $\quad h(0,t) = h_1$ $\quad h(\infty,t) = h_0$

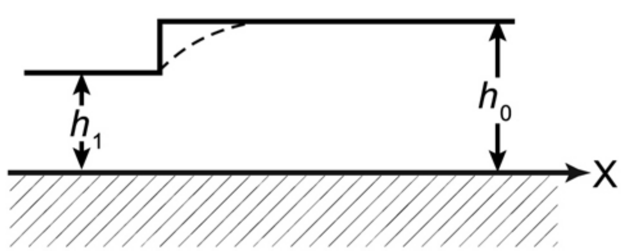

Figure 28.2

Solution is like the solution to the heat flow equation for plate cooling, or the momentum equation for crew shell:

Set $f = \frac{h}{h_0}$ $\quad f(0) = \frac{h_1}{h_0}$

$$\xi = \left(\frac{\eta\phi}{k\rho g h_0 t}\right)^{1/2}\frac{x}{2}$$

$$f(\xi) = f(0)erfc(\xi)$$

Finally, back to the tank demo

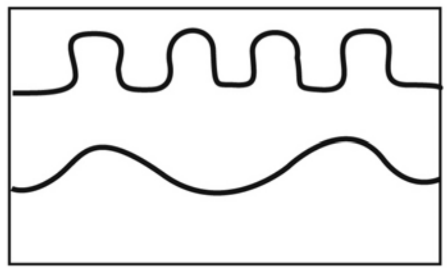

Figure 28.3

Flow between glass plates $\simeq$ Darcy flow.
(recall 1 model of porosity)

Analyze a simple case-1 boundary

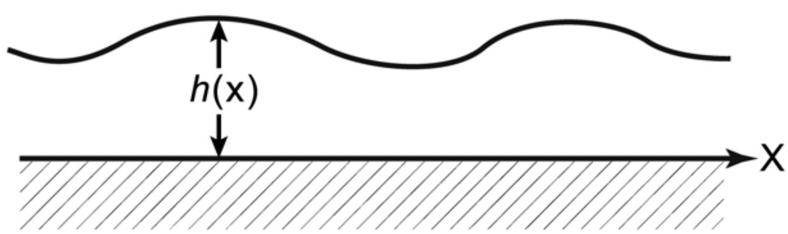

Figure 28.4

To conserve fluid

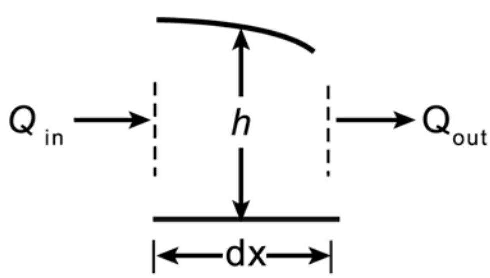

Figure 28.5

$$Q_{out} - Q_{in} = -dx \cdot \frac{dh}{dt} \cdot \phi$$

$$\phi \frac{\partial h}{\partial t} = -\frac{dQ}{dx}$$

but $Q = u \cdot h$, with $u \propto \frac{dh}{dx}$

$$\phi \frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}(uh) = \frac{k\rho g}{\eta} \frac{\partial}{\partial x}(h \frac{\partial h}{\partial x}) \quad \leftarrow \text{nonlinear!}$$

Linearize around $h = h_0 + h_1$

$$\frac{\partial h'}{\partial t} = \frac{k\rho g h_0}{\phi \eta} \frac{\partial^2 h'}{\partial x'^2} \quad \leftarrow \text{diffusion equation}$$

$\neq$ exponential growth or decay

[Note: Dupuit approximation breaks down for large interface perturbation.]

Suppose $h' = \rho \cos kx$, $k = 2\pi / \lambda =$ wave number

$$\frac{\partial^2 h'}{\partial x^2} = -k^2 \rho \cos kx$$

$$\frac{\partial \rho}{\partial t} = -\frac{\mathrm{K} \rho g h_0}{\phi \eta} k^2 \rho$$ where K is permeability.

Exponential growth or decay!

$$\tau = \frac{\eta \phi}{\rho g h_0 \mathrm{K}} \cdot \frac{1}{k^2}$$

How does this compare to Rayleign-Taylor?