12.307 Weather and Climate Laboratory
Spring 2009

12.307
Convection in air (a compressible fluid)

John Marshall, Lodovica Illari and Alan Plumb

March, 2004

1 Dry convection in a compressible atmosphere

Before we can apply intuition from convection in water (an almost incompressible fluid) we must take into account that the atmosphere is a compressible fluid in which $\rho = \rho(p, T)$. In an incompressible fluid $\rho = \rho(T)$ and the stability of the column depends only on the temperature profile (neglecting viscous and diffusive effects); an incompressible fluid is stable to convection if T increases with height, unstable if T decreases with height:

$$\left.\begin{array}{c}\text{UNSTABLE}\\ \text{NEUTRAL}\\ \text{STABLE}\end{array}\right\} \text{ if } \left(\frac{dT}{dz}\right)_E \left\{\begin{array}{l} < 0\\ = 0\\ > 0\end{array}\right. . \tag{1}$$

But let's look at the observed mean $T(z)$ profile of the atmosphere — see Fig.1. We see that T actually *decreases* rapidly in the troposphere and therefore, according to Eq.(1), is statically unstable. In fact the condition stated in Eq.(1) is not appropriate for a compressible fluid like the atmosphere: to figure out whether the atmosphere is 'top heavy' and therefore convectively unstable we have to take into account the effect of p as well as T on ρ.

Let us consider convection in a compressible fluid such as the atmosphere which obeys the perfect gas law, $\rho = p/RT$. The parcel and environmental pressure, temperature and density at $z = z_1$ in Fig.2 are $p_1 = p(z_1)$, $T_1 = T(z_1)$, and $\rho_1 = p_1/RT_1$. The real difference from the incompressible case comes when we consider the adiabatic displacement of the parcel to z_2. As the parcel rises, it moves into an environment of lower pressure. The parcel will adjust to this pressure; in doing so it will expand, do work on its surroundings

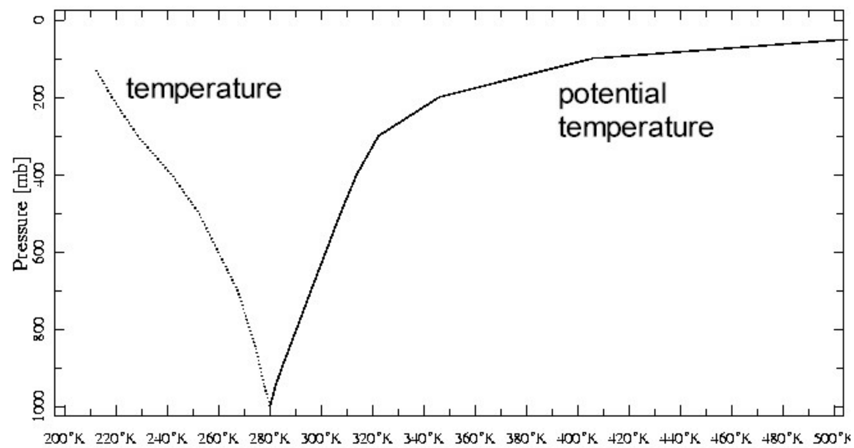

Figure 1: The global average atmospheric T (dotted) and θ (solid) as a function of pressure.

and thus cool. So the *parcel temperature is not conserved* during displacement even if that displacement occurs adiabatically.

In fact there is a second, and very important process, that changes the temperature of rising air parcels. It is common experience that such parcels may become saturated with water vapor and the vapor may condense to form clouds, thereby releasing latent heat into the parcel. We will begin, however, by considering convection in unsaturated air. In order to compute the buoyancy of the parcel in Fig.2 when it arrives at z_2, we need to figure out what happens to its temperature.

1.1 The adiabatic lapse rate (in unsaturated air)

Consider a parcel of ideal gas of unit mass with a volume V, so that $\rho V = 1$. If an amount of heat, δQ, is exchanged by the parcel with its surroundings then applying the first law of thermodynamics (dry air, no latent heat release) gives us:

$$\delta Q = c_v dT + p dV \ , \tag{2}$$

where $c_v dT$ is the change in internal energy due to a change in parcel temperature of dT and pdV is the work done by the parcel on its surroundings by expanding an amount dV. Here c_v is the specific heat at constant volume.

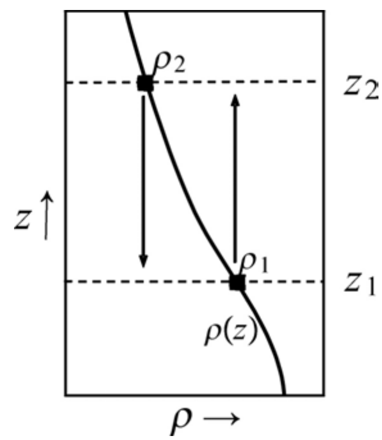

Figure 2: We consider a fluid parcel initially located at height z_1 in an environment whose density is $\rho(z)$. It has density $\rho_1 = \rho(z_1)$, the same as its environment at height z_1. It is now displaced adiabatically a small vertical distance to $z_2 = z_1 + \delta z$ where its density is compared to that of the environment.

Our goal now is to rearrange the rhs of Eq.(2) to express it in terms of dT and dp so that we can deduce how dT depends on dp. To that end we note that, because $\rho V = 1$, $V = \frac{1}{\rho}$; then:

$$dV = d\left(\frac{1}{\rho}\right) = -\frac{1}{\rho^2}d\rho$$

Thus

$$pdV = -\frac{p}{\rho^2}d\rho = -\frac{RT}{\rho}d\rho$$

where, in the last step, we have used the ideal gas law $p = \rho RT$. But, on differentiation, $p = \rho RT$ yields

$$dp = RT\ d\rho + \rho R\ dT\ ,$$

whence

$$p\ dV = -RT\frac{d\rho}{\rho} = -\frac{dp}{\rho} + R\ dT\ .$$

The first law, Eq.(2) can then be written:

$$\begin{aligned}\delta Q &= (R + c_v)\, dT - \frac{dp}{\rho} \\ &= c_p\, dT - \frac{dp}{\rho}\ ,\end{aligned}$$

since $c_p = R + c_v$ where c_p is the specific heat at constant pressure.

For adiabatic motions, $dQ = 0$, whence

$$c_p\, dT = \frac{dp}{\rho}\ . \tag{3}$$

Now if the motions are in hydrostatic balance then $dp = -g\rho_E\, dz$, where ρ_E is the density of the environment (since the parcel and environmental pressures must be locally equal, and the environmental pressure must be in hydrostatic balance with the environmental density). Before being perturbed, the parcel's density was equal to that of the environment. *If the displacement of the parcel is sufficiently small*, its density is still almost equal to that of the environment, $\rho \simeq \rho_E$, and so, under adiabatic displacement, the parcel's temperature will change according to

$$\frac{dT}{dz} = -\frac{g}{c_p} = -\Gamma_d, \tag{4}$$

where Γ_d, the *dry adiabatic lapse rate*, is the rate at which the parcel's temperature decreases with height under adiabatic displacement. Putting in the value of $c_p = 1005 JKg^{-1}K^{-1}$, we find $\Gamma_d \simeq 10\mathrm{K\ km}^{-1}$. Thus, typically, a parcel of dry air displaced vertically cools by $10K$ every km. In order to determine whether or not the parcel experiences a restoring force we must compare its density to that of the environment, as follows.

At z_2, the environment has pressure p_2, temperature $T_2 \simeq T_1 + (dT/dz)_E\, \delta z$, where $(dT/dz)_E$ is the environmental lapse rate, and density $\rho_2 = p_2/RT_2$. The parcel, on the other hand, has pressure p_2, temperature $T_P = T_1 - \Gamma_d \delta z$, and density $\rho_P = p_2/RT_P$. Therefore the parcel will be positively, neutrally, or negatively buoyant according to whether T_P is greater than, equal to, or less than T_1. Thus our stability condition can be written:

$$\left.\begin{array}{c}\text{UNSTABLE}\\ \text{NEUTRAL}\\ \text{STABLE}\end{array}\right\} \text{ if } \left\{\begin{array}{l}\left(\frac{dT}{dz}\right)_E < -\Gamma_d\\ \left(\frac{dT}{dz}\right)_E = -\Gamma_d\\ \left(\frac{dT}{dz}\right)_E > -\Gamma_d\end{array}\right. .$$

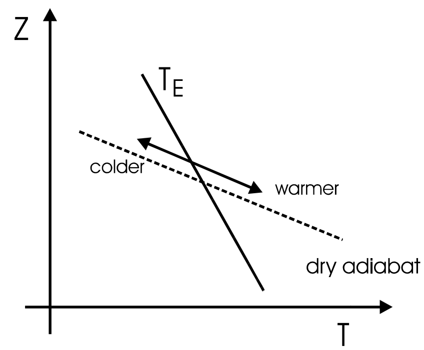

Figure 3: The atmosphere is nearly always stable to dry processes. A parcel displaced upwards (downwards) in an adiabatic process moves along a dry adiabat [the dotted line] and cools down (warms up) at a rate that is faster than that of the environment, $\partial T_E/\partial z$. Since the parcel always has the same pressure as the environment it is not only colder (warmer), but also denser (lighter). The parcel therefore experiences a force pulling it back toward its reference height.

Therefore, a compressible atmosphere is **unstable if temperature decreases with height faster than the adiabatic lapse rate**. This is no longer a simple "top-heavy" criterion (atmospheric density must decrease with height under all conceivable circumstances) nor even a "bottom-warm /top-cool" criterion. Because of the influence of adiabatic expansion, the temperature must decrease with height at greater than the finite rate Γ_d for instability to occur.

The observed global average, lower tropospheric lapse rate — see Fig.1 — is $\left(\frac{dT}{dz}\right)_E \simeq \frac{(T_{500mb}-T_{1000mb})}{(Z_{500mb}-Z_{1000mb})} = \frac{(252.5-280.2)K}{(5.546-0.127)km} \simeq -5K\ km^{-1}$, or about 50% of the adiabatic value. On the basis of our stability results, we would expect no convection, and thus no convective heat transport. In fact the atmosphere is almost always *stable* to dry convection — the situation is as sketched in the schematic, Fig.3. We will see later that it is the release of latent heat when water vapor condenses that leads to convective instability in the troposphere and thus to its ability to transport heat vertically. But before going on we will introduce the very important and useful concept of 'potential temperature', a temperature-like variable that *is* conserved in adiabatic motion.

1.2 Potential temperature

The nonconservation of T under adiabatic displacement makes T a less-than-ideal measure of atmospheric thermodynamics. However, we can identify a quantity called *potential temperature* which *is* conserved under adiabatic displacement.

Using the perfect gas law, $p = \rho RT$, our adiabatic statement Eq.(3) can be rearranged thus:

$$\begin{aligned} c_p dT &= RT\frac{dp}{p} , \\ \frac{dT}{T} &= \frac{R}{c_p}\frac{dp}{p} = \kappa\frac{dp}{p} , \end{aligned}$$

where $\kappa = R/c_p = 2/7$ for a perfect diatomic gas like the atmosphere. Thus

$$d\ln T - \kappa d\ln p = 0$$

and we can define a temperature θ, given by

$$\theta = T\left(\frac{p_0}{p}\right)^{\kappa} \tag{5}$$

which, unlike T, *is* conserved in adiabatic motion — $d\theta = 0$. Here, by convention, we take p_0 to be the constant reference pressure of 1000hPa.

From its definition, Eq.5), we see that θ is the temperature a parcel of air would have if it were expanded or compressed adiabatically from its existing p and T to a standard pressure p_0. It allows one, for example, to directly determine how the temperature of an air parcel will change as it is moved around adiabatically: if we know its θ, all we need to know at any instant is its pressure, and then Eq.(5) allows us to determine its temperature at that instant. For example if the temperature of a parcel of dry air at 300mb is $T = 229K$ (which is $-44°C$), then if it is brought down to the ground adiabatically, its temperature will be $323K$ (or $50°C$). Thus its potential temperature is $\theta = 323K$.

We can express the stability of the column to dry adiabatic processes in terms of θ as follows. Let's return to our displaced air parcel, at the undisturbed position z_1 it has environmental temperature and pressure, and therefore also environmental potential temperature $\theta_1 = \theta_E(z_1)$, where $\theta_E(z)$ is the environmental θ profile. Since the parcel preserves θ, it still has $\theta =$

θ_1 when displaced to z_2. Since its parcel pressure is the same as that of its environment, it is warmer (or cooler) than its environment according to whether θ_1 is greater (or lesser) than $\theta_E(z_2)$. Since $\theta_E(z_2) \simeq \theta_E(z_1) + (d\theta/dz)_E\,\delta z = \theta_1 + (d\theta/dz)_E\,\delta z$, the parcel is

$$\left.\begin{array}{c}\text{UNSTABLE}\\ \text{NEUTRAL}\\ \text{STABLE}\end{array}\right\}\ \text{if}\ \left\{\begin{array}{l}\left(\frac{d\theta}{dz}\right)_E < 0\\ \left(\frac{d\theta}{dz}\right)_E = 0\\ \left(\frac{d\theta}{dz}\right)_E > 0\end{array}\right. \quad , \tag{6}$$

which has the same form as Eq.(1) for an incompressible fluid, but now expressed in terms of θ rather than T. So another way of expressing the instability criterion is that a compressible atmosphere is unstable if **potential temperature decreases with height**.