12.307 Weather and Climate Laboratory
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12.307 Project 1: Radial Inflow Experiment

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Abstract

We rotate a cylinder about its vertical axis: the cylinder has a circular drain hole in the center of its bottom. Water flows inwards, conserving angular momentum and, in so doing, acquires a swirling motion which exhibits a number of important principles of rotating fluid dynamics - conservation of angular momentum, geostrophic and cyclostrophic balance.

1 Introduction

We are all familiar with the swirl and gurgling sound of water flowing down a drain. Here we set up a laboratory illustration of this phenomenon and study it in rotating and non-rotating conditions.

We rotate a cylinder about its vertical axis: the cylinder has a circular drain hole in the center of its bottom. The water flows inward, conserving angular momentum and, in so doing, acquires a swirling motion, as sketched in Fig.1. The swirling motion can become very vigorous if the cylinder is rotated even at only moderate speeds. The swirling flow exhibits a number of important principles of rotating fluid dynamics - conservation of angular momentum, geostrophic (and cyclostrophic) balance, all of which will be studied in detail in this chapter and made use of in our subsequent discussions. The experiment also gives us an opportunity to think about frames of reference. We will in fact use two different versions of this experiment. The first, which was designed by Jack Whitehead of the Woods Hole Oceanographic Institution (for more details refer to Whitehead, J.A and Potter, D.L (1977) Axisymmetric critical withdrawal of a rotating fluid. *Dynamics of Atmospheres and Oceans*, 2, 1-18) uses a recirculating system to produce a stady state. The second, simpler, version simply allows the flow to drain out of the container; the experiment therefore has a finite duration, but works just as well.

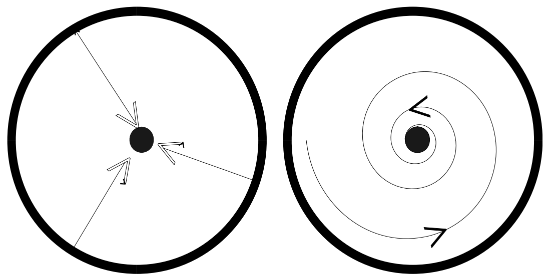

Figure 1: Flow patterns (left) in the absence of rotation and (right) when the apparatus is rotating in an anticlockwise direction.

The main object of our experiment is to measure, and interpret in terms of angular momentum principles, the azimuthal velocity field, $v_\theta(r)$. We will also think about its connection to the pressure field given by the height of the free surface $H(r)$.

2 The apparatus and observed flow patterns

2.1 Recirculating version

We take a cylindrical tank with a drain hole in the center of the bottom - see Fig.2. Water enters at a constant rate through a diffuser on its outer wall and exits through the drain into a reservoir, from which it is pumped back into the diffuser. Thus, a steady state is set up in which the flow down the central drain exactly balances the inflow from the outer edge. The diffuser is effective at producing a symmetrical inflow toward the drain. The entire apparatus is mounted on a turntable and viewed from the laboratory frame and, using a camera mounted above co-rotating with the apparatus, from the rotating frame. The table is turned in an anticlockwise direction (in the same direction as the spinning Earth). The path of fluid parcels is tracked by dropping paper dots on the free surface.

When the apparatus is not rotating, water flows radially inward from the diffuser to the drain in the middle, as sketched in Fig.1 (lhs). The free surface is observed to be rather flat. When the apparatus is rotated, however, the water acquires a swirling motion: fluid parcels *spiral* inward as sketched in Fig.1 (rhs). Even at modest rotation rates of $\Omega = 1$ radian per second (corresponding to a rotation period of around 6 seconds) [1], the effect of rotation is marked and parcels complete many circuits before finally exiting through the drain hole. In the presence of rotation the free surface becomes markedly curved, high at the periphery and plunging downwards toward the hole in the center, as shown in the photograph - see the

[1] Note that if Ω is the rate of rotation of the tank in radians per second, then the period of rotation is $\tau_{\text{tank}} = \frac{2\pi}{\Omega}$. Thus if $\tau_{\text{tank}} = 2\pi s$, then $\Omega = 1$.

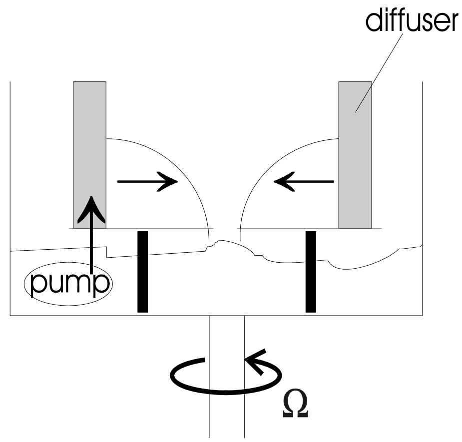

Figure 2: Sketch of radial inflow apparatus. A diffuser with a 30 *cm* inside diameter is constructed of wire screen (and filled with stones approximately 1cm in size), is placed in a larger tank. Water is then fed evenly in to the bottom of the diffuser. The diffuser is effective at producing an axially-symmetric, inward flow at the screen. Below the tank there is a large catch basin, partially filled with water and containing a submersible pump whose purpose is to return the water to the diffuser in the upper tank. The whole apparatus is then placed on a turntable.

photograph in Fig.3.

2.2 Non-recirculating version

In this case, the water is simply allowed to drain away, out of its container. The setup is shown in Fig. 4. A plastic bucket, with a small hole at the center of its base, is mounted inside a sump on a rotating turntable. The bucket should be positioned so that its center coincides with the rotation axis. Initially,the small hole is plugged. The bucket is filled about two-thirds full, and the turntable set into rotation (with rotation rate of around 10 rpm). After the system has spun up, the plug is tapped out of the hole (be careful how you do this – try to do it with the minimum of disturbance to the water in the bucket). As the watr drains away, take measurements of the flow as a function of radius by tracking paper dots on the surface.

2.3 Measurements

Set the tank rotating at a rate of ~ 10 rpm (revolutions per minute). If you are doing the recirculating experiment, turn on the pump and record the flow rate, Q. If you are doing the non-recirculating experiment, allow the water in the bucket to spin up, and tap out the plug. The task is then to measure the surface flow and free surface profiles and compare

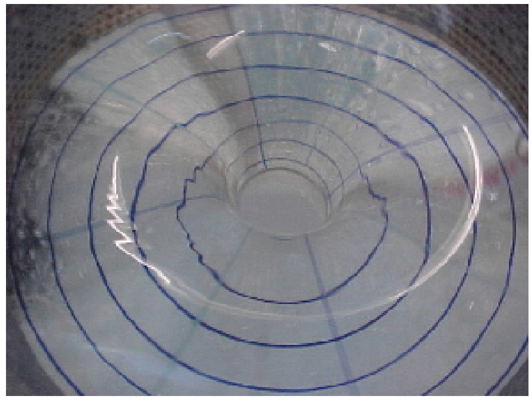

Figure 3: The free surface of the radial inflow experiment viewed in the laboratory frame, in the case when the apparatus is rapidly rotating. The curved surface provides a pressure gradient force directed inwards that is balanced by an outward centrifugal force due to the anticlockwise circulation of the spiraling flow.

Figure 4: The experimental configuration for version 2 of the radial inflow experiment.

them with predictions based on the theory in Section 3. Try various rotation rates and/or flow rates.

1. **Velocity**. Measure the velocity of particles (black paper dots) moving with the flow in the free surface of the fluid. This can be done by recording a sequence of images using the overhead camera and making use of the particle tracking software on the laboratory computers. This will return the coordinates of individual particles as a function of time (frame number). Compute both the azimuthal and radial velocity.

   Check whether the azimuthal speed of the dots, $v_\theta(r)$, is consistent with angular momentum conservation, Eq.(3) below.

   Compute the Rossby number given by Eq.(10). How does it compare to the theoretical prediction, Eq.(12)?

2. **The height field.** If you are doing the recirculating version of the experiment, measure the depth, H, of the water at the radius r_1 and estimate H at other radii. Also record the flow rate through the pump, Q. Use your measurements of radial velocity, v_r, and radial volume flux, Q, to infer $H(r)$ using Eq.(13).

   Are the gradients of the free surface sufficient to balance the centrifugal acceleration in Eq.(5)?

3 Theory

3.1 Dynamical balances

In the limit in which the tank is rotated rapidly, parcels of fluid circulate around many times before falling out through the drain hole; the pressure gradient force directed radially inwards (set up by the free surface tilt) is balanced by a centrifugal force directed radially outwards.

If V_θ is the azimuthal velocity in the absolute frame (the frame of the laboratory) and v_θ is the azimuthal speed *relative* to the tank (measured using the camera co-rotating with the apparatus) then (see Fig. 5):

$$V_\theta = v_\theta + \Omega r \tag{1}$$

where Ω is the rate of rotation of the tank in radians per second. Note that Ωr is the azimuthal speed of a particle fixed to the tank at radius r from the axis of rotation.

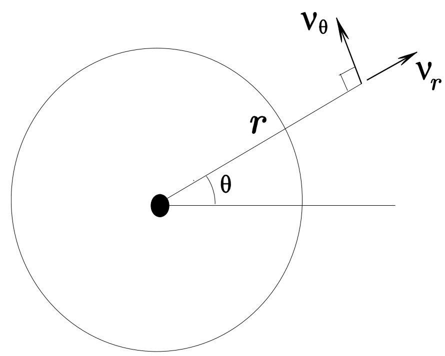

Figure 5: The velocity of a fluid parcel viewed in the rotating frame of reference: $v_{rot} = (v_\theta, v_r)$.

3.1.1 Angular momentum

Fluid entering the tank at the outer wall will have angular momentum because the apparatus is rotating. As parcels of fluid flow inwards axisymmetrically, they will conserve this angular momentum (provided that they are not rubbing against the bottom or the side, so that we may ingore friction). Conservation of angular momentum states that:

$$V_\theta \; r = \text{constant} = \Omega r_1^2 \tag{2}$$

Here r_1 is the inner radius of the diffuser in Fig.(2) and V_θ is the azimuthal velocity in the laboratory (inertial) frame given by Eq.(1). Combining Eqs.(2) and (1) we find:

$$v_\theta = \Omega \frac{(r_1^2 - r^2)}{r} \quad . \tag{3}$$

We now consider the balance of forces in the vertical and radial directions, expressed first in terms of the absolute velocity V_θ and then in terms of the relative velocity v_θ.

3.1.2 Vertical force balance

We suppose that *hydrostatic balance* pertains in the vertical (we'll talk about hydrostatic balance in more detail later.) Then the pressure at height z must be such as to support the fluid above it, so

$$p = \rho g \left(H - z \right) \tag{4}$$

where ρ is the (constant) density, g is the (constant) acceleration due to gravity, $H(r)$ is the height of the free surface and z is a vertical coordinate ($z = 0$ at the base of the tank). Wh here suppose that the pressure vanishes at the free surface (actually $p =$ atmospheric pressure at the surface, which can be taken as zero).

3.1.3 Radial force balance in the non-rotating frame

If the pitch of the spiral traced out by fluid particles is tight (*i.e.* in the limit that $\frac{v_r}{v_\theta} << 1$, appropriate when Ω is sufficiently large[2]) then the centrifugal force directed radially outwards acting on a particle of fluid is balanced by the pressure gradient force directed inwards associated with the tilt of the free surface. This radial force balance can be written in the non-rotating frame thus:

$$\frac{V_\theta^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r}.$$

Using Eq.(4), the radial pressure gradient force in the above can be directly related to the gradient of the free surface enabling the force balance to be written:

$$\frac{V_\theta^2}{r} = g\frac{\partial H}{\partial r} \tag{5}$$

3.1.4 Radial force balance in the rotating frame

Using Eq.(1), we can express the centrifugal acceleration in Eq.(5) in terms of velocities in the rotating frame thus:

$$\frac{V_\theta^2}{r} = \frac{(v_\theta + \Omega r)^2}{r} = \frac{v_\theta^2}{r} + 2\Omega v_\theta + \Omega^2 r \tag{6}$$

Hence

$$\frac{v_\theta^2}{r} + 2\Omega v_\theta + \Omega^2 r = g\frac{\partial H}{\partial r} \tag{7}$$

The above can be simplified by writing $\Omega^2 r = \frac{\partial}{\partial r}\left(\frac{\Omega^2 r^2}{2}\right)$ and defining a quantity h:

$$h = H - \frac{\Omega^2 r^2}{2g}, \tag{8}$$

the height of the free surface measured relative to that of the reference parabolic surface $\frac{\Omega^2 r^2}{2g}$ (see notes on 'parabolic surface'). Then Eq.(7) can be written in term of h thus:

$$2\Omega v_\theta + \frac{v_\theta^2}{r} = g\frac{\partial h}{\partial r} : \qquad \text{gradient wind balance} \tag{9}$$

Eq.(5) and Eq.(9) are completely equivalent statements of the balance of forces. The distinction between them is that the former is expressed in terms of V_θ, the latter in terms

[2] Note that this assumption is relaxed in the appendix.

of v_θ. Note that Eq.(9) has the same form as Eq.(5) except an extra term, $-2\Omega v_\theta$, appears on the rhs of Eq.(9) - this is called the 'Coriolis acceleration'. It has appeared because we have chosen to express our force balance in terms of *relative*, rather than absolute velocities.

Let us compare the magnitude of the $\frac{v_\theta^2}{r}$ and $2\Omega v_\theta$ terms in Eq.(9). Their ratio is the 'Rossby number':

$$R_o = \frac{|v_\theta|}{2\Omega r} \tag{10}$$

If $R_o << 1$, the $\frac{v_\theta^2}{r}$ term can be neglected in (9). In this limit, Coriolis and pressure gradient terms balance one another.

$$2\Omega v_\theta = g\frac{\partial h}{\partial r} : \quad \text{geostrophic balance} \tag{11}$$

Equation (11) is a simple form of the 'geostrophic equation' relating velocities in the rotating frame to the horizontal pressure gradient in the limit of small R_o. So, how large is R_o in our experiment? We can estimate its size by computing v_θ based on angular momentum conservation.Using the angular momentum conserving prediction (3) in (10) we find, for this profile,

$$R_o = \frac{1}{2}\left(\frac{r_1}{r}\right)^2 - 1. \tag{12}$$

Thus $R_o = 1$ at $r = r_1/\sqrt{3}$; $R_o < 1$ if $r > r_1/\sqrt{3}$ (the region of geostrophic balance) and so, in the outer regions of the flow, the inward radial pressure gradient is balanced by outward Coriolis forces (small R_o): the flow is in geostrophic balance here. In the inner regions, $r < r_1/\sqrt{3}$, $R_o > 1$. But as parcels spiral into the drain they pass through a region where R_o becomes increasingly large and v_θ^2/r in Eq.(9) becomes a dominant term. In the limiting case $R_o \gg 1$, the centrifugal term dominates the Coriolis term in Eq.(9), and we have what is known as "cyclostrophic balance:"

$$\frac{v_\theta^2}{r} = g\frac{\partial h}{\partial r} : \quad \text{cyclostrophic balance}$$

3.2 Mass balance

If the volume source coming radially inwards through the diffuser has strength Q, then conservation of volume tells us that:

$$2\pi r H V_r = Q \tag{13}$$

where $V_r(r)$ is the radial velocity at radius r.

4 Appendix

Eq.(5) is an approximate statement of radial force balance in the limit that the pitch of the spiral traced out by fluid particles is tight (*i.e.* in the limit that $\frac{v_r}{v_\theta} << 1$). If this is not true then we must include radial accelerations and use the following more accurate statement of radial momentum equation:

$$\underbrace{V_r \frac{\partial V_r}{\partial r}}_{\text{radial acc}^{\text{n}}} - \underbrace{\frac{V_\theta^2}{r}}_{\text{centrifugal acc}^{\text{n}}} = \underbrace{-g \frac{\partial H}{\partial r}}_{\text{pressure gradient}} \tag{14}$$

Here V_r is the radial component of velocity.

4.0.1 Solutions

Using Eqs.(2) and (13), Eq.(14) can be written:

$$\frac{\partial}{\partial r}\left(\frac{Q^2}{8\pi^2 r^2 H^2}\right) - \frac{\Omega^2 r_1^4}{r^3} = -g\frac{\partial H}{\partial r}$$

which, on integration, can be written:

$$gH + \frac{Q^2}{8\pi^2 r^2 H^2} + \frac{\Omega^2 r_1^4}{2r^2} = \text{constant} \tag{15}$$

where H_1 is the depth of the water at radius r_1. Eq.(15) is a cubic for H which can be solved.

Use your velocity measurements of estimate how large is the term $V_r \frac{\partial V_r}{\partial r}$ relative to $\frac{V_\theta^2}{r}$ in Eq.(14) in your experiment.

Use (13) to show that if

$$\frac{Q}{2\pi H \Omega r_1^2} << 1 \tag{16}$$

then the force balance Eq.(14) reduces to Eq.(5). Can you see that Eq.(16) is just the condition that $\frac{v_r}{v_\theta} << 1$?