12.510 Introduction to Seismology
Spring 2008

12.510 Introduction to Seismology
2/20/08

Stress and Strain

Reading: Section 2.3 of Stein and Wysession

In this lecture, we will examine using Newton's 2nd Law and a generalized form of Hooke's Law to characterize the response of a continuous medium to applied forces

The <u>stress tensor</u> describes the forces acting on internal surfaces of a deformable, continuous medium.

The <u>strain tensor</u> describes the distortion of (or the variation in displacement within) the body

In order to produce wavemotion, we need
- a spatial change in stress
- a way to describe the causal relationship between stress and strain

In 1-D, for a purely elastic medium, this relationship turns out to be quite simple:

$$\boxed{\sigma = E\varepsilon}$$

where σ is stress, ε is strain, and E is Young's modulus (which must have dimensions of stress, as strain is dimensionless.

The assumption of a purely elastic medium is one that we will make most of the time in this class. The relationships between stress and strain are dependent on the characteristics of the medium.

Sidebar: Another possibility occurs for a viscoelastic medium:

$$\boxed{\sigma = \zeta\dot{e}}$$

where σ is stress, $\dot{e}$ is strain rate ($\delta\varepsilon/\delta t$), and ζ is viscosity.

In three dimensions, things become somewhat more complicated. A simple example should convince that the straightforward linear relationship given above can no longer hold. If we stretch a rubber band along its long axis (in this case applying a σ_x)…

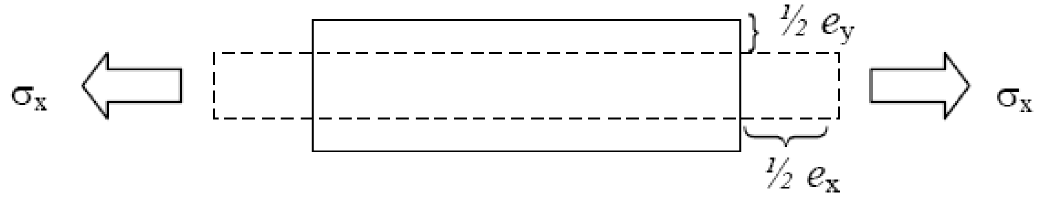


Figure 1

We can describe the strain along the x-axis in the normal way,

$$\varepsilon_x = \frac{l + \Delta l}{l}$$

but we can see that despite the fact that $\sigma_y = 0$, ε_y does not equal zero (and a similar case could be made for the z direction) so we must turn to tensors to formulate our Hooke's Law for three dimensions.

We can represent the stress tensor in the form σ_{ij} , where I represents the direction of the normal vector to the surface upon which the stress acts, and j represents the direction of the stress. For three dimensions, each of these can be broken down into x, y, and z components. Consider the volume element below.

Figure 2.3-2: Traction vectors on the faces of a volume element.

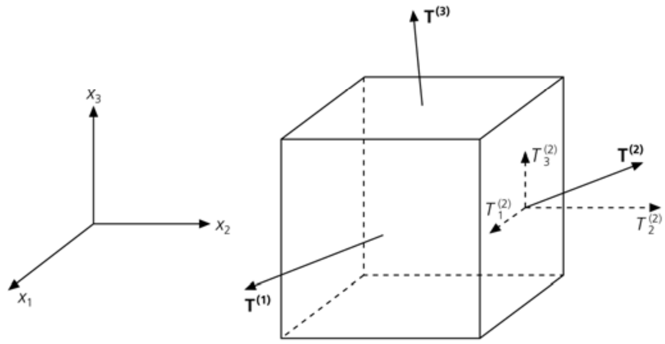


Fig. 2 (After *Stein & Wyssession*, 2003)

Strain can be represented by a similar tensor, in the form $\varepsilon_{kl.}$ The relationship between stress and strain must therefore take the form of a 4th order tensor that we call the elasticity (or stiffness) tensor. The elasticity tensor gives us information about the medium.

We can now write Hooke's Law in a general form.

$$\boxed{\sigma_{ij} = c_{ijkl}\varepsilon_{kl}}$$

Later, when we look at the Lamé parameters, remember that they must reside somewhere in the beast $c_{ijkl.}$

This equation becomes very complex if it turns out that the elements of c are dependent on stress or strain (i.e. non-linear), but for small deformations we assume that this is not the case.

So far we have made the following assumptions (a list that will expand as we proceed)

Assumption #1: Our medium is purely elastic

12.510 Introduction to Seismology
2/20/08

Assumption #2: The elements of c are independent of stress and strain

Next, we will develop Newton's second law towards our goal of expressing an equation of motion.

Newton's second law simply states:

$$\boxed{\sum F_i = ma_i}$$

Starting with the left hand side of the equation, we submit that the applicable forces fall neatly into one of two categories:

1. <u>Body Forces</u>: forces such as gravity that work equally well on all particles within the mass- the net force is proportional (essentially) to the volume of the body (f_i).
2. <u>Surface Forces</u>: forces that act on the surface of a body- the net force is proportional to the surface area over which the force acts.

The surface force $\mathbf{F}_s$ acts on a surface element dS which has a unit normal vector **n**. The forces acting on the surfaces of a volume element can be described by three mutually perpendicular traction vectors using the same convention that we described earlier for the stress tensor on a volume element. We define the traction as the limit of the surface force per unit area at any point as the area becomes infinitesimal

$$Traction \equiv \vec{T} = \frac{\lim}{dS \Rightarrow 0} \frac{\vec{F}}{\delta S} = (T_1, T_2, T_3)$$

and recall from our volume element (this time using x_1, x_2, and x_3 instead of x, y, and z for coordinate axes) that each traction can be thought of as the sum of forces acting upon an infinitesimal surface, broken down into normal and tangent components, such that

$$\vec{T}_1 = (T_{11}, T_{12}, T_{13})$$

$$\vec{T}_2 = (T_{21}, T_{22}, T_{23})$$

$$\vec{T}_3 = (T_{31}, T_{32}, T_{33})$$

An equivalent notation used in the textbook (Stein and Wysession) gives

$$\vec{T}_1 = (T_1^{(1)}, T_2^{(1)}, T_3^{(1)}) \quad \text{, etc.}$$

where the upper index indicates the surface normal and the lower index gives the force component direction.

12.510 Introduction to Seismology
2/20/08

This set of nine terms that describes the surface forces can be grouped into the stress tensor σ_{ji}. The tensor's rows are comprised of the three traction tensors, presented below in such a way as to demonstrate the equivalence of various methods of notation.

$$\sigma_{ji} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix} = \begin{bmatrix} T_1^{(1)} & T_2^{(1)} & T_3^{(1)} \\ T_1^{(2)} & T_2^{(2)} & T_3^{(2)} \\ T_1^{(3)} & T_2^{(3)} & T_3^{(3)} \end{bmatrix}$$

Backing up a bit, we return to our discussion of Newton's 2nd Law, this time separating out body forces (f_i) from surface forces (T_i)

$$\sum F_i = ma_i$$

which we can expand to:

$$\int_V f_i dV + \int_S T_i dS = ma_i$$

Looking at this equation, we see that we are adding a surface integral to a volume integral, so we should want to convert one to the other form. Since we have mass on the right hand side (which is an integration of density over volume), we should elect to convert the surface integral into a volume integral, which can be done using the Gauss (Divergence) Theorem.

For the time being, we will leave this idea to simmer, but will return to collect it when we develop equations of motion at the end of this lecture.

Before we continue, let's pause for a moment to collect our bearings:

If we look again at the stresses, we recall that we have defined them as a force per unit area, so we can think of them as similar to pressures, with units of Pa.

$$\boxed{1 \text{ atm} = 1 \text{ bar} = 1000 \text{ mbar} = 10^5 \text{ Pa}}$$

Strain is defined as a change in some dimension (length, volume, angle) over that same dimension, so it is by definition dimensionless. This means that our elasticity/ stiffness tensor c_{ijkl} must also have dimensions of stress (Pa).

To develop a feeling for some typical values, we present a brief table of values for lithostatic stress at various depths.

Lithostatic stress ≡ The normal stress due to the weight of the overlying rock (overburden)

Depth	Typical values for lithostatic stress
410 km	14 GPa
660 km	23 GPa
CMB	130 GPa
Center of the Earth	360 GPa

We can see that these values are quite extreme. Actually, for seismic propagation we are more interested specifically in non-lithostatic stresses, but more on that later.

The stress tensor gives a traction vector acting on an arbitrary surface element within the medium. We can show this by presenting the stress tensor on an infinitesimal tetrahedron, where one face is a surface element with a normal vector **n** that is not aligned with any coordinate axis. A traction is applied to this surface. The other three faces of the tetrahedron are each perpendicular to a coordinate axis and to each other.

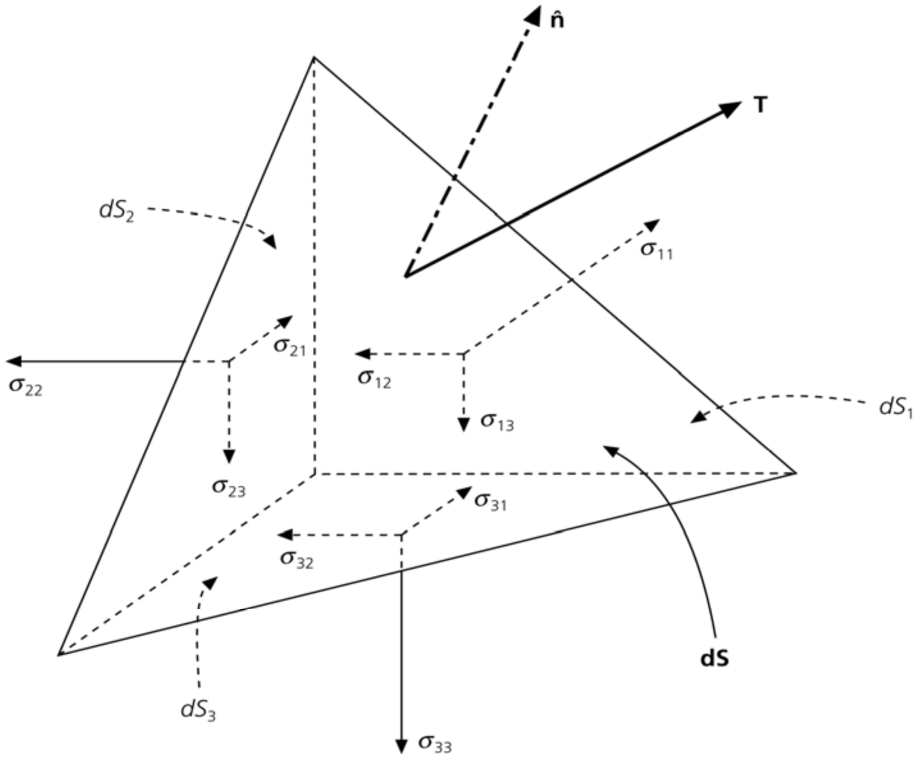


Fig. 3 (After *Stein & Wyssession,* 2003)

We can decompose the traction into components acting on our perpendicular surfaces.

$\vec{T} = (T_1, T_2, T_3)$

Note that in doing this we have implicitly made a third assumption, namely that

Assumption #3: Volume elements within the medium can be treated as continuums

Each of our T_i also have three components (a total of nine elements), so we can rewrite the stress tensor as:

$$\boxed{T_i dS = \sigma_{1i}\Delta S_1 + \sigma_{2i}\Delta S_2 + \sigma_{3i}\Delta S_3}$$

We can also see that there is a relationship between angles (and surfaces), specifically that $\Delta S_i = \cos\gamma_i dS$, where γ_i is the angle between the normal vector and the ith coordinate axis. This means that we can rewrite our stress tensor as a sum of scalar products.

$$\boxed{T_i = \sigma_{1i} \cdot \hat{n}_1 + \sigma_{2i} \cdot \hat{n}_2 + \sigma_{3i} \cdot \hat{n}_3}$$

or in Einstein summation

$$\boxed{T_i = \sigma_{ji} \cdot \hat{n}_j}$$

Likewise we can look at the elements of the stress tensor

$$\boxed{\sigma_{ji} = T_i^{(j)}}$$

In the absence of body forces, the stress tensor is treated as symmetric ($\sigma_{ij}=\sigma_{ji}$), so there are only six independent elements. The diagonal elements represent the normal stress and the off-diagonal elements the shear stress.

Assumption #4: The stress tensor is symmetric ($\sigma_{ij}=\sigma_{ji}$)

A symmetric tensor can also be rotated into a principal coordinate frame such that the tractions become parallel to the normals (i.e. the shear stresses go to zero) In other words,

$$T_i = \sigma_{ji} \cdot \hat{n}_j = \sigma_{ij} \cdot \hat{n}_j = \lambda \hat{n}_i$$

which is a straightforward eigenvalue/eigenvector problem.

$$(\sigma_{ij} - \lambda\delta_{ij})\hat{n}_j = 0$$

12.510 Introduction to Seismology
2/20/08

$$\det\begin{bmatrix} \sigma_{11}-\lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\lambda \end{bmatrix} = 0$$

Solving the determinant for zero gives a cubic equation, therefore three solutions for λ (eigenvalues) which can be plugged back in to get the principal stress axes (eigenvectors).

$$\sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

In seismology, we typically order these such that $\sigma_1 > \sigma_2 > \sigma_3$ (i.e. σ_1 is the most compressive stress).

Some situations in which the sigma values have specific relationships are of particular interest.

1. Uniaxial stress ($\sigma_1 \neq 0$, $\sigma_2 = \sigma_3 = 0$)

2. Plane stress ($\sigma_1 \neq 0$, $\sigma_2 = 0$, $\sigma_3 \neq 0$)

3. Pure shear stress ($\sigma_1 = -\sigma_3$, $\sigma_2 = 0$)
   (This is actually a special case of plane stress)

4 Isotropic stress ($\sigma_1 = \sigma_2 = \sigma_3 = P$)
   (also termed lithostatic/hydrostatic stress)

As we stated earlier, in seismic propagation, we are interested in the non-lithostatic stresses, i.e. that which is left over when we subtract out the lithostatic stress.

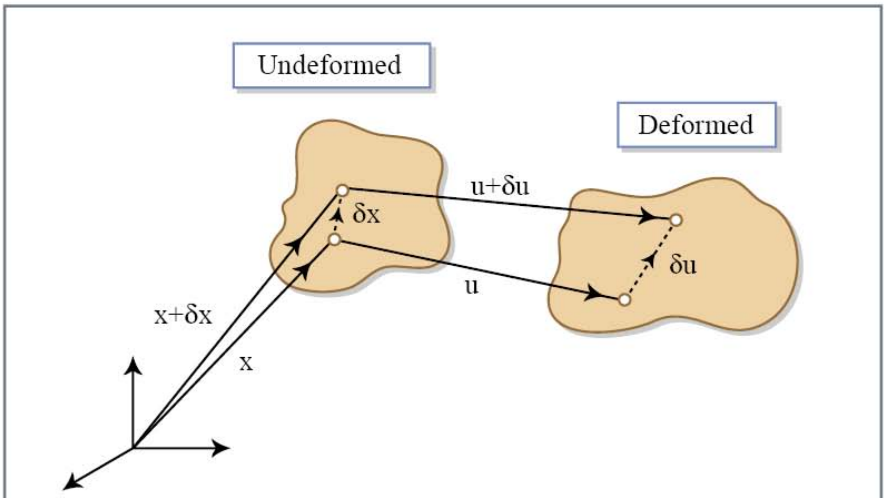
Deviatoric Stress ≡ the remaining stress after the effect of the mean stress ($P=1/3\ (\sigma_1 + \sigma_2 + \sigma_3)$) has been removed.

$$\sigma'_{ij} = \begin{bmatrix} \sigma_{11}-P & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-P & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-P \end{bmatrix}$$

12.510 Introduction to Seismology
2/20/08

Strain

When stress is applied to a non-rigid body deformation occurs. This deformation can be described by the strain tensor. Strain is a relative measurement and is therefore dimensionless.



Change in relative displacement during deformation.

Figure by MIT OCW.

(Adapted from *Stein & Wyssession*, 2003)

In 1D the strain is given by: $\varepsilon_{xx} = \frac{u(x+\delta x)-u(x)}{\delta x} = \frac{\delta u}{\delta x} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right)$, where we used the linearization: $u(x+\delta x) \cong u(x) + \delta u(x)$. This is justified as long as the change in displacement is smooth over a distance δx:

$\delta u \approx \int_x \frac{\partial u}{\partial x}\delta x$ (infinitesimal strain theory).

$$\mathbf{u}(x) = \mathbf{u}(x_0) + \frac{\partial u}{\partial x}\mathbf{d}$$

$$\mathbf{u}(x) = \mathbf{u}(x_0) + \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \\ & & \\ & etc & \\ & & \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \mathbf{u}(x_0) + \mathbf{J}\mathbf{d}$$

Where **J** is the Jacobian transformation tensor.

$$\mathbf{J} = \boldsymbol{\varepsilon} + \boldsymbol{\Omega}$$

$\boldsymbol{\varepsilon}$ is the symmetric matrix strain tensor ε_{ij}

$$e_{ij} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{pmatrix}$$

$\mathbf{\Omega}$ is the antisymmetric matrix rotation tensor Ω_{ij}

$$\Omega_{ij} = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial u_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & 0 & \frac{1}{2}(\ ) \\ -\frac{1}{2}(\ ) & -\frac{1}{2}(\ ) & 0 \end{pmatrix}$$

In seismology we are interested only in the distortion of the material (strain tensor) and not the rigid body rotation (rotation tensor).

The trace (tr) of the strain tensor is

$tr(\boldsymbol{\varepsilon}) = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = \nabla.\mathbf{u}$, which is also known as the cubic dilatation (Θ).

Divergence of the displacement field relates to the relative change in volume. The trace of the rotation vector is zero, i.e. a rigid body rotation does not involve a volume change.

More on C_{ijkl}:

This 4th-rank tensor makes things complicated because it has 81 elements, so we must use symmetry to simplify C_{ijlk}. Because of symmetry in σ,

(10) $C_{ijkl} = C_{jikl}$

Because of symmetry in $\boldsymbol{E}$,

(11) $C_{ijkl} = C_{ijlk} = C_{jilk}$

Therefore, we have reduced C_{ijkl} to 36 independent elements. Another way to look at it is that there are 6 independent elements of σ as well as 6 independent elements of $\boldsymbol{E}$, giving 36 independent elements to C_{ijkl}. Furthermore, one can also demonstrate using the idea of strain energy that

(12) $C_{ijkl} = C_{klij}$,

further reducing the tensor to 21 independent elements.

Unfortunately, displacement can be observed in at most 3 directions, meaning that only 3 observations are available to determine 21 unknowns. We need to make further assumptions to reduce the number of unknowns. Let's assume that we are working in an ***isotropic medium***, i.e. the medium has the same physical properties in all directions.

12.510 Introduction to Seismology
2/20/08

Isotropy allows C_{ijkl} to be expressed in only two independent terms, giving

(13) $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$,

where λ and μ are ***Lamé parameters*** (named for Gabriel Lamé, a 19th century French mathematician) and δ_{ij}, δ_{kl}, etc. are ***Kronecker symbols*** such that

(14) $\delta_{ij} = 0$ when $i \neq j$

(15) $\delta_{ij} = 1$ when $i = j$

As a consequence of the linear behavior of stress and strain, λ and μ are not dependent on strain. μ refers to the ***rigidity***, or ***shear modulus***, of the medium. μ measures the resistance against shear, i.e. an easily sheared material has a smaller μ than one that is difficult to shear. For a fluid, $\mu = 0$.

λ does not have a specific physical characteristic to make it intuitively understandable. It is defined with respect to the shear modulus using the ***bulk modulus***, K, such that

(16) $K = \lambda + 2\mu/3$

The bulk modulus measures a material's incompressibility, a relative change in volume (cubic dilatation) Δ due to change in pressure P, such that

$K = -\delta P/\delta\Delta$

A material that is hard to compress or has a smaller relative volume will give a higher bulk modulus than a material that is easy to compress or has a larger relative volume.

A note on dimensions:
Because strain is dimensionless, C_{ijkl} must have units of stress. Thus, λ and μ are given in Pascal.

Using equation (13), Young's modulus can be written in terms of Lamé's parameters for an isotropic medium:

(17) $E = \sigma_{xx}/E_{xx} = (3\lambda+2\mu)\mu \;/\; (\lambda+\mu)$

Additionally, we often see ***Poisson's ratio*** ν, where

(18) $\nu = \lambda \,/\, 2(\lambda+\mu)$

Poisson's ratio is often used to characterize the elastic properties of a medium, for example, for a fluid with μ=0, ν=0.5 . As the rigidity of a material increases to infinity, Poisson's ratio approaches 0.

A ***Poisson's medium*** is an isotropic material with Lamé parameters such that $\lambda = \mu$, giving ν=0.25 . This value of Poisson's ratio is reasonable for many crustal and mantle rocks, so it is often assumed in calculations. In the inner core, $\nu = 0.4$, suggesting that the inner core is more "mushy" and sponge-like than the mantle, but still maintains some rigidity.

When considering P- and S-waves in the crust and mantle, we can assume a Poisson's medium in order to relate them, such that

(19) $V_p \cong V_s \sqrt{3}$

Now, expanding σ_{ij} for an isotropic medium with perfect linear elasticity, we get

$$(20) \qquad \sigma_{ij} = C_{ijkl}e_{kl} = \begin{bmatrix} \lambda\Delta + 2\mu e_{11} & 2\mu e_{12} & 2\mu e_{13} \\ 2\mu e_{12} & \lambda\Delta + 2\mu e_{22} & 2\mu e_{23} \\ 2\mu e_{13} & 2\mu e_{23} & \lambda\Delta + 2\mu e_{33} \end{bmatrix}$$

Notice that the off-diagonals are pure shear stresses, and they are only dependent on μ. The diagonals refer to the normal stress and depend on both μ, λ, and Δ (i.e, change in volume).

12.510 Introduction to Seismology
2/20/08

Aside:
Generic anisotropy takes us back to 21 unknowns in C_{ijkl}, so we have to make assumptions about the symmetry of the medium. For instance, in an olivine-rich medium, such as the mantle, the olivine crystals tend to align themselves in a constant direction. Thus, seismic waves will propagate faster along the crystal alignment than in other directions, creating anisotropy and necessitating 5 independent elements in C_{ijkl} (hexagonal symmetry; transverse isotropy).

Equation of Motion

Let us revisit the stress tetrahedron. We have a traction T that can be broken up into 3 components, (T_1, T_2, T_3). We also know, using Newton's 2nd Law of Motion and a force balance on the tetrahedron, that

(21) $\Sigma F = T_i \delta S - (\sigma_{i1} n_1 \delta S + \sigma_{i2} n_2 \delta S + \sigma_{i3} n_3 \delta S) + f_i dV = ma = \rho\,(\delta^2 u_i / \delta t^2)\, dV$

where $f_i dV$ represents the body forces on the tetrahedron. Ignoring the body forces and assuming a=0, this gives us

(22) $T_i = \sigma_{ij} n_j$,

which is true for pure equilibrium.

Now, consider an accelerating seismic wave. The equation of motion will be

(23) $(T_i - \sigma_{ij} n_j)\delta S + f_i dV = \rho\,(\delta^2 u_i / \delta t^2)\, dV$

If the traction cancel, i.e., $T_i - \sigma_{ij} n_j$, equation (22) would give

(24) $f_i = \rho\,(\delta^2 u_i / \delta t^2)$, that is acceleration would only be due to the body forces;

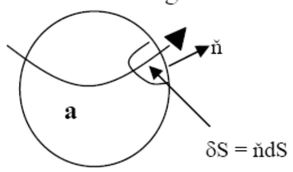
but if there is a net change in stress, $(T_i - \sigma_{ij} n_j)$ can be considered as the non-lithostatic (deviatoric) stress, σ_{ij}'. Therefore,

(25) $\sigma_{ij}' u_j \delta S + f_i dV = \rho\,(\delta^2 u_i / \delta t^2)\, dV$

From now on, σ_{ij} will be used to refer to the deviatoric stress, σ_{ij}'.

Now, to develop equation (25) further we want to get rid of either δS or dV. We will use ***Gauss' Divergence Theorem*** to transform a surface integral into a volume integral. Gauss' theorem uses flux to relate volume to surface area.

Consider a field **a** with a flux through a surface with area δS:



The total flux of the field in and out of the surface is given by

(26) $\int \mathbf{a} . \mathbf{dS} = \int a_i n_i dS$

which is related to the amount of field generated or absorbed in the volume within dS. In other words

(27) $_S\int a_i n_i dS = \int \nabla \cdot \mathbf{a}\, dV$,

where $\nabla \cdot \mathbf{a}$ is the source (or sink) of the field. This is Gauss' Divergence Theorem. If $\nabla \cdot \mathbf{a} = 0$, the field is source/sink-free.

12.510 Introduction to Seismology
2/20/08

So, taking equation (25) and integrating both sides, we get

(28) $\int \sigma_{ij} u_j \delta S + \int f_i dV = \int \rho\, (\delta^2 u_i/\delta t^2)\, dV$

Now we apply Gauss' Theorem and combine terms to get

(29) $\int (\delta\sigma_{ij}/\delta x_j + f_i) dV = \int \rho\, (\delta^2 u_i/\delta t^2)\, dV$,

which leads to

(30) $\rho\, (\delta^2 u_i/\delta t^2) = \delta\sigma_{ij}/\delta x_j + f_i = \sigma_{ij,\, j} + f_i$ (Stein and Wysession not.) $= \delta_j \sigma_{ij} + f_i$ (van der Hilst not.)

In vector form, we get the equation of motion,

(31) $\rho \ddot{\mathbf{u}} = \boldsymbol{f} + \nabla \cdot \sigma$

Equation (30 or 31) is known as either ***Navier's*** or ***Cauchy's Equation of Motion***, as they independently worked on developing it in the 19th century.

updated by Shane