Lecture 12

12.1 Administration

- Mid term on the 30th:
  - We haven't done much on the applied front, so the exam will likely be dominated by comprehension-like questions.
  - May be asked to derive and/or explain bit of equations.
  - I'll try hard to make it an interesting exam, and not just ask you to regurgitate what you've been shown in class.
  - Emphasis will be on understanding what you've been shown, not just remembering what you've been shown.
- $\mathbf{u} = \nabla\phi$ versus $\mathbf{u} = -\nabla\phi$
- **QUIZ!**

12.2 Back to the potential (or point) vortex

If a number of potential vortices are placed in a 2D inviscid fluid, the vortices will mutually interact and advect one another around.

(Draw a cartoon here).

We will call these 2D potential vorticies "point vortices." We have 2D, inviscid, irrotational flow (potential flow), so we know that the evolution of the system is governed by the linear Laplace equations. Thus, the principal of superposition applies, and one can think of the flow as that which is produced by adding together the impact of each point vortex in turn. Consider a single point vortex. It just sits there. Consider two point vortices of equal strength but opposite signs (see figure 12.1). They will advect each other in the same direction. This can be shown with a little experiment using a tank of water, dye, and a ruler (as seen in figure 12.2).

(Perform experiment).

The two point vortices will stay aligned, propagate together, and will be heavily modified by viscosity. We can quantify this. u_θ from vortex A at the location of vortex B will be $u_\theta(B) = -\frac{\Gamma}{2\pi\Delta x}$. Thus, for $\theta = 0$ (as we have for B with respect to A) we are left with

$$u_x = -u_\theta \sin\theta = 0 \tag{12.1}$$

$$u_y = u_\theta \cos\theta = -\frac{\Gamma}{2\pi\Delta x} \tag{12.2}$$

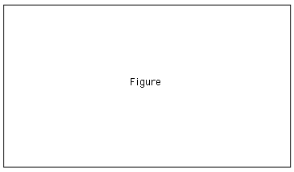

Figure 12.1: (`fig:Lec12TwoPointVortices`) With two point vortices of equal strength but opposite signs, they advect one another down the page **(Q1)**.

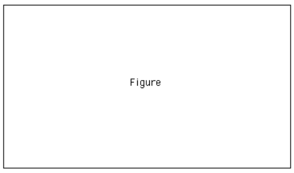

Figure 12.2: (`fig:Lec12TwoPointVorticesExperiment`) By pulling (or pushing) a fluid with a ruler, two point vortices of opposite sign and nearly equal strength are generated. They will stay roughly aligned and propagate together.

So, vortex A will move vortex B in the negative y-direction at a speed of $\frac{\Gamma}{2\pi\Delta x}$. u_θ from vortex B at the location of vortex A will be $u_\theta(A) = \frac{\Gamma}{2\pi\Delta x}$. Thus, for $\theta = \pi$ (as we have for A with respect to B) we have

$$u_x \;=\; -u_\theta \sin\theta \;=\; 0 \qquad (12.3)$$

$$u_y \;=\; u_\theta \cos\theta \;=\; -\frac{\Gamma}{2\pi\Delta x} \qquad (12.4)$$

So, vortex B will have vortex A moving in the negative y-direction at a speed $\frac{\Gamma}{2\pi\Delta x}$. Thus, vortices A and B will remain Δx apart and propagate in the negative y-direction at a speed of $\frac{\Gamma}{2\pi\Delta x}$.

What happens to our vortices if a wall gets in the way?

(Perform experiment).

The vortices propagated as before , but as they got close to the walls they spread apart. We can quantify this using the *method of images.* This is a remarkably common trick when dealing with systems that are governed by Laplace's equation. The idea is that instead of just inserting a wall, you mirror each vortex on the other side of a notional wall – see figure 12.3. In this picture, B is the actual vortex, but it will propagate in the negative y-direction at a speed $\frac{\Gamma}{2\pi\Delta x}$.

If there are 2 vortices, we will have a situation as is shown in figure 12.4. A and B are "real" and C and D are "mirrored" such that A and B are as far from the wall as C and D, and

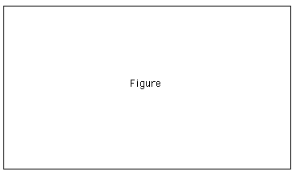

Figure 12.3: (`fig:Lec12MethodOfImages1`) Dealing with vortices and the presence of a wall by using the method of images. The actual vortex, B, will propagate in negative y-direction at a speed $\frac{\Gamma}{2\pi\Delta x}$, same as for the solution with two vorticies.

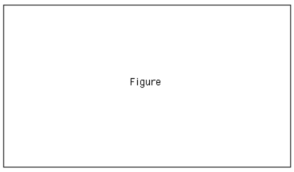

Figure 12.4: (`fig:Lec12MethodOfImages2`) Applying the method of images to the situation where there are two vortices approaching a wall. Here A and B are real and C and D are mirrored. A and B propagate towards the wall together, and then split, leaving on their respective sides.

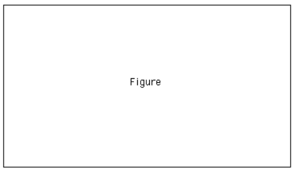

Figure 12.5: (`fig:Lec12VelocityContributionsOnA`) Velocity contributions on A from the other (real and mirrored) point vorticies. Note that the velocity of vortex A has a component from vortex B (negative y-direction, a component from vortex C (negative x-direction), and a component from vortex D (combination of positive x and y-directions). You just add the influence of each of the vorticies. The resulting effect is that A moves toward the wall, and the away from B along the wall.

$\Gamma_B = \Gamma_C = -\Gamma_A = -\Gamma_D$. A and B will propagate towards the wall, C and D will also propagate towards the wall, but C on A will move A in the positive x-direction while D on A will move A in the positive x and y-directions. See the velocity contributions on A as seen in figure 12.5.

Your problem set for the next week asks you to write some point vortex code. How the heck are you going to do that? Remember, a point vortex only moves because of the velocity field generated by other point vortices. Once we know the velocity of a given point vortex, we can change its position based to the velocity and a time increment. There are many ways to do this, and I would suggest using the fourth order Runga-Kutta scheme you downloaded to produce the plot of the Lorenz attractor. You'll need to feed `stepit` a new time that takes a vector that contains the current point vortex locations and returns their velocities.

Since the vector will contain x and y positions, the velocities will need to be in x and y coordinates. How do we get them? We know how to get u_θ velocities (see figure 12.6). Break u_θ into u and v:

$$u = -u_\theta \sin\theta \tag{12.5}$$

$$v = u_\theta \cos\theta \tag{12.6}$$

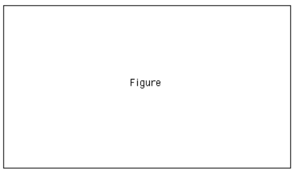

Figure 12.6: (`fig:Lec12FindUTheta`) Setup for finding velocities in terms of x and y coordinates from u_θ.

But from the geometry between point vortices we know that (This works for all θ):

$$\sin\theta = \frac{\Delta y}{r},\ \cos\theta = \frac{\Delta y}{r} \tag{12.7}$$

$$\begin{aligned}
\Rightarrow u &= -u_\theta \frac{\Delta y}{r} = -\frac{\Gamma}{2\pi}\frac{\Delta y}{r^2} && (12.8)\\
\Rightarrow v &= u_\theta \frac{\Delta x}{r} = \frac{\Gamma}{2\pi}\frac{\Delta x}{r^2} && (12.9)
\end{aligned}$$

You sum up the u's and v's for all point vortices acting on the point of interest to get the resultant u and v that ends up moving the point.

12.3 Kelvin's Circulation Theorem

Now we will move towards writing the momentum equation in terms of vorticity. We'll start by asking the question: how does the circulation of a fluid blob change with time?

$$\Gamma = \int_c \mathbf{u}\cdot d\mathbf{x} = \iint_A (\nabla\times\mathbf{u})\cdot\ d\mathbf{A} \tag{12.10}$$

This is a Lagrangian blob so we write:

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\oint \mathbf{u}\cdot d\mathbf{x} = \oint_c \frac{d}{dt}(\mathbf{u}\cdot d\mathbf{x}) \tag{12.11}$$

Note according for change in $\mathbf{u}$ and change in $d\mathbf{x}$ **(Jim - What do you mean by this?)**.

$$\begin{aligned}
\frac{d\Gamma}{dt} &= \oint \mathbf{u}\frac{d(dx)}{dt} + \oint d\mathbf{x}\cdot\frac{d\mathbf{u}}{dt} && (12.12)\\
&= \oint_c \mathbf{u}\frac{d(dx)}{dt} + \oint_c \frac{d\mathbf{u}}{dt}\cdot d\mathbf{x} && (12.13)\\
&= \oint_c \mathbf{u}\cdot d\mathbf{u} + \oint_c \frac{d\mathbf{u}}{dt}\cdot d\mathbf{x} && (12.14)\\
&= \oint_c d\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) + \oint_c \frac{d\mathbf{u}}{dt}\cdot d\mathbf{x} && (12.15)\\
&= 0 + \oint_c \frac{d\mathbf{u}}{dt}\cdot d\mathbf{x} && (12.16)\\
\frac{d\Gamma}{dt} &= \oint_c \frac{d\mathbf{u}}{dt}\cdot d\mathbf{x} && (12.17)
\end{aligned}$$

Note that the term going to zero is because the integral is around a closed loop. That's a Langrangian $\frac{d}{dt}$, so let's switch to an Eulerian way of writing it:

$$\frac{d\Gamma}{dt} = \oint_c \frac{D\mathbf{u}}{Dt}\cdot d\mathbf{x} \tag{12.18}$$

That $\frac{D\mathbf{u}}{Dt}$ should look very familiar from the momentum equations:

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \frac{1}{\rho}\mu\nabla^2\mathbf{u} + \frac{1}{\rho}\frac{\mu}{3}\nabla(\nabla\cdot\mathbf{u}) \tag{12.19}$$

Assume an inviscid fluid:

$$\frac{d\Gamma}{dt} = \oint_c \left(-\frac{1}{\rho}\nabla p + \mathbf{g}\right) \cdot d\mathbf{x} \tag{12.20}$$

Recall from when we derived the Bernoulli equations:

$$\mathbf{g} = -\nabla gz \tag{12.21}$$

Also recall that if we say $\rho = \rho(p)$:

$$-\frac{1}{\rho}\nabla p = -\nabla \int \frac{dp}{\rho} \tag{12.22}$$

Plugging in

$$\frac{d\Gamma}{dt} = \int_c \left(-\nabla \int \frac{dp}{\rho} - \nabla gz\right) \cdot d\mathbf{x} \tag{12.23}$$

$$= \int_c \nabla \left(-\int \frac{dp}{\rho} - gz\right) \cdot d\mathbf{x} \tag{12.24}$$

$$= \int_c d\left(-\int \frac{dp}{\rho} - gz\right) \tag{12.25}$$

$$\frac{d\Gamma}{dt} = 0 \tag{12.26}$$

Because of the closed contour of integration. This is true if the fluid is:

- Inviscid
- $\rho = \rho(p) \Rightarrow$ barotropic $\Rightarrow p = p(\rho)$
- Only conservative body forces

12.4 Where Kelvin's Circulation Theorem breaks down

1. How does this ($\frac{d\Gamma}{dt} = 0$) break down if viscosity is thrown back into the mix?

   Vorticity is generated on boundaries ($\mathbf{u} = 0$ at boundary, $\mathbf{u} = \mathbf{u}$ away from boundary). If your closed contour moves into these boundary layers, then vorticity is diffused onto the circuit and the circulation changes.

2. How does $\frac{d\Gamma}{dt} = 0$ break down if we have non-barotropic flow?

   Barotropic $\Rightarrow p = p(\rho)$ or $\rho = \rho(p)$. Consider the situation where you have fluid at rest and then heat at a point at the bottom of the fluid. In this situation $\rho = \rho(p, T)$, so we shouldn't expect Kelvin's Theorem to hold. How does it break down? See figures 12.7.

   Kelvin's theorem breaks down because of the non-barotropic (baroclinic) effects. Generically, the typical pictures that are drawn are as seen in figures 12.8 and 12.9.

3. How does $\frac{d\Gamma}{dt} = 0$ break down if we have non-conservative body forces?

   Non-conservative forces are almost always things like the friction force, so Kelvin's first criterion sorts them out. Kundu suggests that a conservative force acts through the center of mass of a blob but it isn't obvious to me that *all* non-conservative forces don't act through the center of mass. But if is safe to say that if we have a body force that does not operate thorugh the center of mass, then vorticity will be generated, $\frac{d\Gamma}{dt} \neq 0$

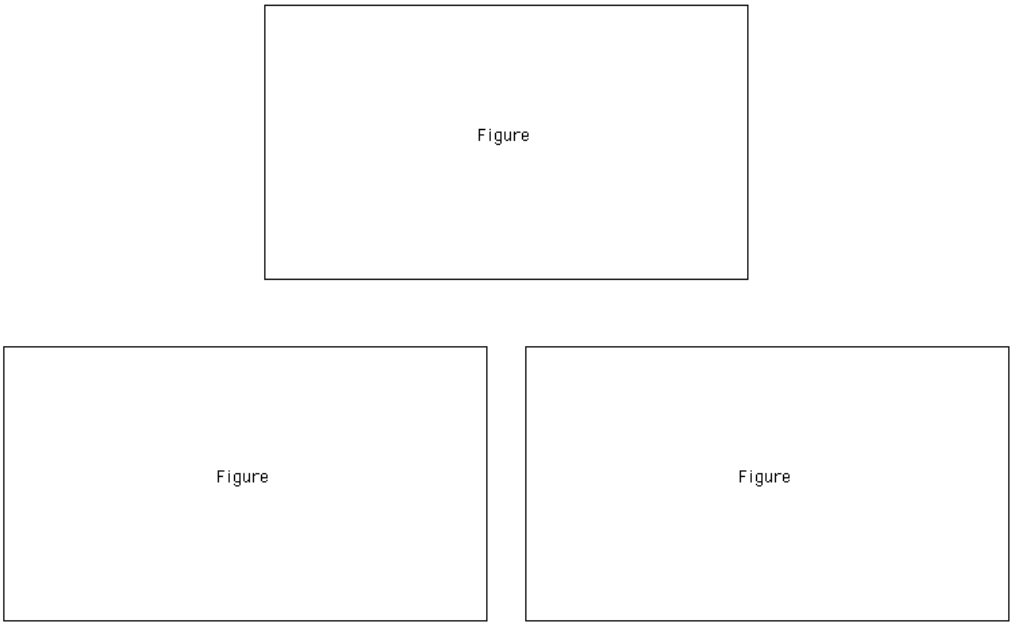

Figure 12.7: (`fig:Lec12Convection, fig:Lec12BeforeHeating, and fig:Lec12AfterHeating`) Warmer less dense fluid rises, cools and then sinks (see top panel). The lower left panel is before heating, the panel on the lower right left is after.

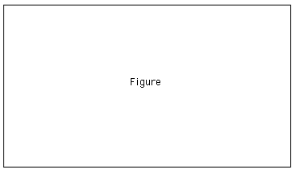

Figure 12.8: (`fig:Lec12Barotropic`) Barotropic: When $p = p(\rho)$ ($\rho = \rho(p)$)lines of constant ρ and lines of constant p are parallel. Net pressure forces act through the center of mass and there is no tendency for the blob to develop rotation about its center of mass.

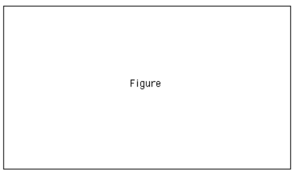

Figure 12.9: (`fig:Lec12Baroclinic`) Baroclinic: When density is functionally dependent on properties other than only pressure we have a baroclinic situation. Here lines of constant ρ are not necessarily parallel to lines of constant p. The net pressure gradient force need not pass through the center of mass of the blob and if it doesn't, there will be a torque about the center of mass, fluid will tend to rotate about the center of mass, and vorticity will be generated.

12.5 Reading for Class 13

KCO1: 4.12
CR: 2.1-2.5