Lecture 8

8.1 Administration

- Collect PS4.

- Distribute PS5 (due October 14th, maybe 16th).

- No class next Tuesday.

- Greg Lawson next Thursday.

8.2 Constitutive relationship

Ended last time with an expression for stress

$$\tau = 2\mu\mathbf{e} - \left(p + \frac{2}{3}\mu\nabla\cdot\mathbf{u}\right)\mathbf{I} \tag{8.1}$$

We got here by assuming a linear relationship

$$\tau = a\mathbf{e} + b\mathbf{I} \tag{8.2}$$

and stated $a = 2\mu$. Hence

$$\tau = 2\mu\mathbf{e} + b\mathbf{I} \tag{8.3}$$

This was argued to be the most simple linear expression that allowed stress in the absence of motion

$$\text{no motion} \quad \Rightarrow \quad \tau = b\mathbf{I} = -p \tag{8.4}$$

Before we plug this expression for stress into the momentum equation, let's spend a little time trying to figure out what it means

$$\begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} = \begin{bmatrix} 2\mu\frac{\partial u}{\partial x} & \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & 2\mu\frac{\partial v}{\partial y} & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & 2\mu\frac{\partial w}{\partial z} \end{bmatrix}$$
$$- \begin{bmatrix} p + \frac{2}{3}\mu\nabla\cdot\mathbf{u} & 0 & 0 \\ 0 & p + \frac{2}{3}\mu\nabla\cdot\mathbf{u} & 0 \\ 0 & 0 & p + \frac{2}{3}\mu\nabla\cdot\mathbf{u} \end{bmatrix} \tag{8.5}$$

First, look at one of the off-diagonal (shear stress) terms:

$$\tau_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{8.6}$$
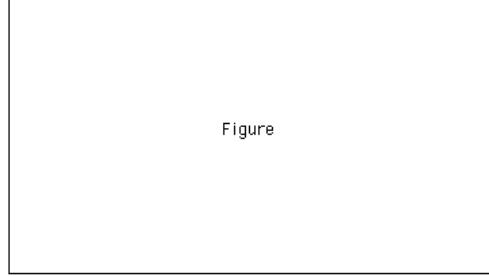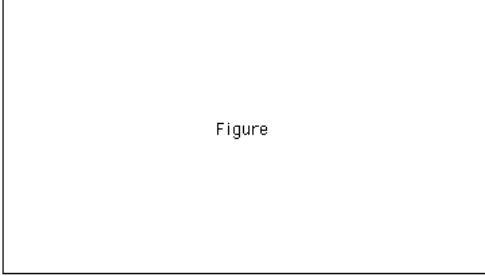
Figure 8.1: (`fig:Lec8Tauxx1`, `fig:Lec8Tauxx2`) For the case in the left panel, there is no contribution to the stress term; everything balances. For the case in the right panel, there is a contribution from the bit of $\frac{\partial u}{\partial x}$ that is greater than the average divergence.

Nothing more than the Hooke's Law expression we came up with in our 2nd and 3rd lecture. The prop between applied force and rate of deformation.

Now for the diagonal terms:

$$
\begin{aligned}
\tau_{xx} &= 2\mu\frac{\partial u}{\partial x} - p - \frac{2}{3}\mu\nabla\cdot\mathbf{u} \\
&= -p + 2\mu\left(\frac{\partial u}{\partial x} - \frac{1}{3}\nabla\cdot\mathbf{u}\right) \\
&= -p + 2\mu\left(\frac{\partial u}{\partial x} - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right)
\end{aligned}
\tag{8.7}
$$

$$
\tag{8.8}
$$

There's a pressure term, and the second term says the normal strain rate only contributes to the stress if it is larger than the average of the normal strain rates. See figure 8.1.

8.3   Momentum equation

Now let's plug the constitutive relationship (8.1) into the momentum equation:

$$
\begin{aligned}
\rho\frac{D\mathbf{u}}{Dt} &= \rho\mathbf{g} + \nabla\cdot\tau \\
\Rightarrow \quad \rho\frac{D\mathbf{u}}{Dt} &= \rho\mathbf{g} + \nabla\cdot\left(2\mu\mathbf{e} - \left(p + \frac{2}{3}\mu\nabla\cdot\mathbf{u}\right)\mathbf{I}\right)
\end{aligned}
\tag{8.9}
$$

This is straight forward in Einstein notation, but a bit uglier in vector notation. For example:

$$
\text{Einstein notation} \quad \Rightarrow \quad \nabla\cdot p\mathbf{I} = \sum_{i=1}^{3}\delta_i\left(\sum_{j=1}^{3}\frac{\partial}{\partial x_j}pI_{ij}\right), \quad I_{ij} = 0, i\neq j \tag{8.10}
$$

$$
\begin{aligned}
\text{Vector notation} \quad \Rightarrow \quad \nabla\cdot p\mathbf{I} &= \frac{\partial}{\partial x}p\,\mathbf{i} + \frac{\partial}{\partial p}p\,\mathbf{j} + \frac{\partial}{\partial z}p\,\mathbf{k} \\
&= \nabla p
\end{aligned}
\tag{8.11}
$$

Now what?

$$
\nabla\cdot\left[\begin{array}{ccc} 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\nabla\cdot\mathbf{u} & \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ \vdots & \vdots & \vdots \end{array}\right]
\tag{8.12}
$$

Look at the **i** component of the resulting dot product:

$$
\left[\frac{\partial}{\partial x}\left(2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu \nabla \cdot \mathbf{u} \right) + \frac{\partial}{\partial y}\mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z}\mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] \mathbf{i}
$$

$$
= \left[\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial x^2} - \frac{2}{3}\mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z} \right) + \mu \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial^2 v}{\partial x \partial y} + \mu \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z} \right] \mathbf{i}
$$

$$
= \left[\mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \frac{1}{3}\mu \left(\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z} \right) \right] \mathbf{i}
$$

$$
= \left[\mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \frac{1}{3}\mu \frac{\partial}{\partial x}\nabla \cdot \mathbf{u} \right] \mathbf{i} \tag{8.13}
$$

The other two directions will give

$$
+ \quad \left[\mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \frac{1}{3}\mu \frac{\partial}{\partial y}\nabla \cdot \mathbf{u} \right] \mathbf{j} \tag{8.14}
$$

$$
+ \quad \left[\mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \frac{1}{3}\mu \frac{\partial}{\partial z}\nabla \cdot \mathbf{u} \right] \mathbf{k} \tag{8.15}
$$

Thus, in vector notation, this all gives

$$
\nabla \cdot \left(2\mu \mathbf{e} - \frac{2}{3}\mu \nabla \cdot \mathbf{u}\,\mathbf{I} \right) \quad = \quad \mu \nabla^2 \mathbf{u} + \frac{1}{3}\mu \nabla(\nabla \cdot \mathbf{u}) \tag{8.16}
$$

Substituting this last expression into the momentum equation, we get

$$
\rho \frac{D\mathbf{u}}{Dt} \quad = \quad \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{u} + \frac{1}{3}\mu \nabla(\nabla \cdot \mathbf{u}) \tag{8.17}
$$

This is the Navier-Stokes equation! Here, the $-\nabla p$ term comes from normal stress, the $\mu \nabla^2 \mathbf{u}$ term mostly from shear, and the $\frac{\mu}{3}\nabla(\nabla \cdot \mathbf{u})$ term from the divergence. Note that while the Navier-Stokes equation is help up as the gold standard, the viscosity stuff is full of assumptions and μ is empirically obtained. The following are simplifications that are sometimes made.

Incompressible: Assuming incompressibility implies $\nabla \cdot \mathbf{u} = 0$.

$$
\rho \frac{D\mathbf{u}}{Dt} \quad = \quad \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{u} \tag{8.18}
$$

Ignore viscosity:

$$
\rho \frac{D\mathbf{u}}{Dt} \quad = \quad \rho \mathbf{g} - \nabla p \tag{8.19}
$$

This is also known as the Euler equation, a perfect fluid, or an ideal fluid. More on this later.

8.4   Hydrostatic relation

Consider the full momentum equation

$$
\rho \frac{D\mathbf{u}}{Dt} \quad = \quad \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{u} + \frac{1}{3}\mu \nabla(\nabla \cdot \mathbf{u}) \tag{8.20}
$$

What does it look like with *static flow*, $\mathbf{u} = 0$?

$$
0 \quad = \quad \rho \mathbf{g} - \nabla p \tag{8.21}
$$

$$
\Rightarrow \quad \frac{\partial p}{\partial x} = 0, \qquad \frac{\partial p}{\partial y} = 0, \qquad 0 = -\rho g - \frac{\partial p}{\partial z} \tag{8.22}
$$

This last equation, $\frac{\partial p}{\partial z} = -\rho g$, is the hydrostatic relation. It tells us how pressure varies with density and height. We'll shall need it soon.

8.5 Momentum equation and the Boussinesq approximation

Again, start with the full momentum equation

$$\rho\frac{D\mathbf{u}}{Dt} \;=\; \rho\mathbf{g} - \nabla p + \mu\nabla^2\mathbf{u} + \frac{1}{3}\mu\nabla(\nabla\cdot\mathbf{u}) \tag{8.23}$$

Boussinesq continuity says $\nabla\cdot\mathbf{u} = 0$

$$\Rightarrow \quad \rho\frac{D\mathbf{u}}{Dt} \;=\; \rho\mathbf{g} - \nabla p + \mu\nabla^2\mathbf{u} \tag{8.24}$$

This should upset you! While we derived the Boussinesq continuity relationship earlier, it does not necessarily mean that this result can be directly substituted into the momentum equation. Another scaling argument must be made with all of the terms in the momentum equation. While it works out to be the same as a substitution in this case, it will not be worked out here in the interest of time.

Boussinesq also says $\rho = \rho_o + \rho'(x, y, z, t)$ where $\rho_o \gg \rho'$. This implies $p = p_o(z) + p'(x, y, z, t)$ where $p_o(z) \gg p'$. $p_o(z)$ goes with the ρ_o part of density and obeys the hydrostatic balance (ρ' terms drop from scaling):

$$\frac{\partial p_o}{\partial z} \;=\; -\rho_o g \tag{8.25}$$

Now let's look at the momentum equations by velocity component, and make further scaling arguments:

$$x: \qquad \rho\frac{Du}{Dt} \;=\; -\frac{\partial p}{\partial x} + \mu\nabla^2 u$$

$$(\rho_o + \rho')\frac{Du}{Dt} \;=\; -\frac{\partial}{\partial x}(p_o + p') + \mu\nabla^2 u$$

Since $\rho_o \gg \rho'$ and p_o is not a function of x we are left with

$$x: \qquad \rho_o\frac{Du}{Dt} \;=\; -\frac{\partial p'}{\partial x} + \mu\nabla^2 u \tag{8.26}$$

Similarly for y:

$$y: \qquad \rho_o\frac{Dv}{Dt} \;=\; -\frac{\partial p'}{\partial y} + \mu\nabla^2 v \tag{8.27}$$

Next, z:

$$z: \qquad \rho\frac{Dw}{Dt} \;=\; -\rho g - \frac{\partial p}{\partial x} + \mu\nabla^2 w$$

$$(\rho_o + \rho')\frac{Dw}{Dt} \;=\; -(\rho_o + \rho')g - \frac{\partial}{\partial z}(p_o + p') + \mu\nabla^2 w$$

$$\rho_o\frac{Dw}{Dt} \;=\; -\rho_o g - \rho' g - \frac{\partial p_o}{\partial z} - \frac{\partial p'}{\partial z} + \mu\nabla^2 w$$

Here, the ρ' is again argued to be small compared to ρ_o in the RHS, but *not* in the LHS. (**Jim — Perhaps this warrants further explanation).** Next we substitute in the hydrostatic relation, which says $\frac{\partial p_o}{\partial z} = -\rho_o g$, yielding a cancellation of terms:

$$\rho_o\frac{Dw}{Dt} \;=\; -\rho' g - \frac{\partial p'}{\partial z} + \mu\nabla^2 w$$

But why doesn't the $-\rho' g$ term go away as "small"?

1. One can argue that you need it physically (ρg is the weight of the fluid). (**Jim $-$ But this is just the perturbation density; isn't $\rho_o g$ more realistically the weight of the fluid?**).

2. Can argue that since there is no associated $\rho_o g$ term, it isn't smaller tat something else. (**Jim $-$ But there was earlier in the derivation; in fact we made use of $\rho_o \gg \rho'$ once, but selectively chose not to do it elsewhere which allows for the presence of the $\rho'g$ term here at the end**).

3. Because it is not small relative to $\rho_o \frac{Dw}{Dt} \Rightarrow \frac{Dw}{Dt}$ must be small, and it is! In fact, when we get to GFD it can be argued that $\frac{Dw}{Dt} = 0$! (**Jim $-$ But didn't we already assume $\frac{Dw}{Dt} = 0$ to get to the hydrostatic relation?**).

Put this all into vector notation and we end up with

$$\rho_o \frac{D\mathbf{u}}{Dt} = \rho'\mathbf{g} - \nabla p' + \mu\nabla^2\mathbf{u} \tag{8.28}$$

It is typical to drop the primes and write

$$\frac{D\mathbf{u}}{Dt} = \frac{\rho'}{\rho_o}\mathbf{g} - \frac{1}{\rho_o}\nabla p' + \nu\nabla^2\mathbf{u} \tag{8.29}$$

Here $\nu = \frac{\mu}{\rho_o} \equiv$ kinematic viscosity. (**Jim $-$ Needless to say, I found this z component part a little confusing; especially looking at from the point of view of someone learning fluids and scaling for the first time**).

8.6   The 1st Law

I would really like to concentrate on the momentum equations, but so long as we're in the swing of derivations, let keep at it. Back in lecture 3 we wrote

$$\frac{dE}{dt} = \dot{Q} - \dot{W} \tag{8.30}$$
$$E \equiv \text{total energy}$$
$$\dot{Q} \equiv \text{rate of heat transfer; } + \text{when added to system}$$
$$\dot{W} \equiv \text{rate of work; } + \text{when done by system}$$

Here, E is our extensive property. The associated intensive property will be made up of two terms:

$$e \equiv \text{internal energy per unit mass}$$
$$\frac{1}{2}\mathbf{u} \cdot \mathbf{u} = \frac{1}{2}\mathbf{u}^2 \equiv \text{kinetic energy per unit mass}$$

Thus, the intensive property is $\rho(e + \frac{1}{2}\mathbf{u}^2) \equiv$ energy/volume, where $de = c_v dT, c_v \equiv$ specific heat at constant volume. c_v is the ratio of the amount of heat, Q, it takes to raise a mass, m, by an amount ΔT. $c_v = Q/m\Delta T$. RTT says that

$$\frac{dE}{dT} = \iiint_V \rho \frac{D}{Dt}\left(e + \frac{1}{2}\mathbf{u}\right) dV = \dot{Q} - \dot{W} \tag{8.31}$$

Recall this? When an intensive property is a product of ρ, the constant equation wipes out the $\rho(e + \frac{1}{2}\mathbf{u}^2)\nabla \cdot \mathbf{u}$ term.

Now consider $\dot{Q}$, the heat transfer bit. We want it in integral form, and it's the heat transfer across the boundaries of our blob, so it will be a surface integral.

$$\dot{Q} \;=\; \iint\limits_{A} \mathbf{q} \cdot d\mathbf{A} \tag{8.32}$$

But what sign to give it? $\dot{Q}$ is positive when work is added to the system, but something fluxing into a control volume has a negative sign since $d\mathbf{A}$ is positive outward.

$$\dot{Q} \;=\; -\iint\limits_{A} \mathbf{q} \cdot d\mathbf{A} \tag{8.33}$$

Now for the $\dot{W}$ term. Work is force times distance, $W = \mathbf{F} \cdot \Delta\mathbf{x}$. So the work rate will be force times distance divided by time, $\dot{W} = \mathbf{F} \cdot \frac{\Delta\mathbf{x}}{\Delta t}$, or force times velocity

$$\dot{W} \;=\; \mathbf{F} \cdot \mathbf{u} \tag{8.34}$$

Recall from our derivation of the momentum equations, that the forces acting on our blob of fluid are body and surface forces:

$$\mathbf{F} \;=\; \mathbf{F}_B + \mathbf{F}_S \tag{8.35}$$

This ignores the line force, which is surface tension. Thus

$$\dot{W} \;=\; \mathbf{F}_B \cdot \mathbf{u} + \mathbf{F}_S \cdot \mathbf{u} \tag{8.36}$$

From our work on the momentum equations, the only body force that we are considering so far, is gravity. It acts on all the mass in the blob, so we have the volume integral

$$\mathbf{F}_B \cdot \mathbf{u} \;=\; \iiint\limits_{V} \rho\,\mathbf{g} \cdot \mathbf{u}\, dV \tag{8.37}$$

Note that $\dot{W} > 0$ when work is done *by* the system, and since $\mathbf{g}$ is *on* the system, $\mathbf{F}_B \cdot \mathbf{u} < 0$. Hence we use

$$\mathbf{F}_B \cdot \mathbf{u} \;=\; -\iiint\limits_{V} \rho\,\mathbf{g} \cdot \mathbf{u}\, dV \tag{8.38}$$

Again, from our work on momentum, we went through a lot of pain to convince ourselves that the surface force was given by the stress tensor. It's force on the surface of the blob, so it will be an area integral

$$\mathbf{F}_S \cdot \mathbf{u} \;=\; \iint\limits_{A} (\tau \cdot \mathbf{u}) \cdot d\mathbf{A} \tag{8.39}$$

$\dot{W} > 0$ when work is done *by* the system, but here we have work being done *on* the system, so

$$\mathbf{F}_S \cdot \mathbf{u} \;=\; -\iint\limits_{A} (\tau \cdot \mathbf{u}) \cdot d\mathbf{A} \tag{8.40}$$

(Jim – It might be less confusing if every $\dot{Q}$ and $\dot{W}$ component is presented only once; immediately with the appropriate sign).

We not have all the terms for the energy equation (8.30):

$$\frac{dE}{dt} \;=\; \dot{Q} - \dot{W}$$

$$\iiint\limits_{V} \rho \frac{D}{Dt}\left(e + \tfrac{1}{2}\mathbf{u}\right) dV \;=\; -\iint\limits_{A} \mathbf{q} \cdot d\mathbf{A} - \left[-\iiint\limits_{V} \rho\,\mathbf{g} \cdot \mathbf{u}\, dV - \iint\limits_{A} (\tau \cdot \mathbf{u}) \cdot d\mathbf{A} \right]$$

$$\iiint\limits_{V} \rho \frac{D}{Dt}\left(e + \tfrac{1}{2}\mathbf{u}\right) dV \;=\; \iiint\limits_{V} \rho\,\mathbf{g} \cdot \mathbf{u}\, dV + \iint\limits_{A} (\tau \cdot \mathbf{u}) \cdot d\mathbf{A} - \iint\limits_{A} \mathbf{q} \cdot d\mathbf{A} \tag{8.41}$$

Now we need to convert the two area integrals on the RHS into volume integrals using the divergence theorem:

$$\iint_A (\tau \cdot \mathbf{u}) \cdot d\mathbf{A} \quad = \quad \iiint_V \nabla \cdot (\tau \cdot \mathbf{u}) dV \qquad (8.42)$$

$$-\iint_A \mathbf{q} \cdot d\mathbf{A} \quad = \quad -\iiint_V \nabla \cdot \mathbf{q} \, dV \qquad (8.43)$$

Inserting these results in the energy equation (8.41) yields

$$\iiint_V \rho \frac{D}{Dt}\left(e + \frac{1}{2}\mathbf{u}\right) dV \quad = \quad \iiint_V \rho \mathbf{g} \cdot \mathbf{u} \, dV + \iiint_V \nabla \cdot (\tau \cdot \mathbf{u}) dV - \iiint_V \nabla \cdot \mathbf{q} \, dV$$

$$\iiint_V \rho \frac{D}{Dt}\left(e + \frac{1}{2}\mathbf{u}\right) dV \quad = \quad \iiint_V \left[\rho \mathbf{g} \cdot \mathbf{u} + \nabla \cdot (\tau \cdot \mathbf{u}) - \nabla \cdot \mathbf{q}\right] dV \qquad (8.44)$$

Since this integral must hold for all volumes

$$\rho \frac{D}{Dt}\left(e + \frac{1}{2}\mathbf{u}\right) \quad = \quad \rho \mathbf{g} \cdot \mathbf{u} + \nabla \cdot (\tau \cdot \mathbf{u}) - \nabla \cdot \mathbf{q} \qquad (8.45)$$

This is the energy equation. Note rising motion leads to an increase in potential energy as $\mathbf{g} \cdot \mathbf{u} < 0$, as it should since it is on the RHS. The terms:

$$\rho \mathbf{g} \cdot \mathbf{u} \quad \equiv \quad \text{changes in potential energy}$$
$$-\nabla \cdot \mathbf{q} \quad \equiv \quad \text{changes due to heat flux}$$
$$\nabla \cdot (\tau \cdot \mathbf{u}) \quad \equiv \quad \text{has a kinetic energy part and an internal heating part}$$

Let's look into the term containing stress a bit more deeply:

$$\nabla \cdot (\tau \cdot \mathbf{u}) \quad = \quad \mathbf{u} \cdot (\nabla \cdot \tau) + \tau : \nabla \mathbf{u} \qquad (8.46)$$

Here the first term, $\mathbf{u} \cdot (\nabla \cdot \tau)$, is the rate of work done by the surface forces. Imbalances accelerate the fluid blob and change its kinetic energy (scaler). The second term contains a double contraction, $\tau : \nabla \mathbf{u}$, which is a scalar. This is the work done by deformation. Not all stress moves the blob. Some of it deforms and causes heat change.

8.7 Reading for class 9

KC01: 4.16 - 4.17