Boussinesq Approximation

There are several versions of the Boussinesq approximation around; the general purpose is to isolate the important effects of density — the buoyancy — and eliminate the less important ones (altering horizontal pressure forces, sound waves). We begin with the full Euler equations

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}p_x$$
$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}p_y$$
$$\frac{Dw}{Dt} = -\frac{1}{\rho}p_z - g$$
$$\frac{D\rho}{Dt} + \rho(u_x + v_y + w_z) = 0$$
$$\frac{D\rho}{Dt} - \frac{1}{c_s^2}\frac{Dp}{Dt} = \frac{\rho_T Q}{c_p}$$

and define a hydrostatic state

$$\frac{\partial\overline{p}}{\partial z} = -g\overline{\rho}$$

and deviations from this state,

$$p = \overline{p} + \overline{\rho}\varphi$$
$$\rho = \overline{\rho}(1+\sigma)$$

Then our momentum equations become

$$\frac{Du}{Dt} - fv = -\frac{1}{1+\sigma}\varphi_x$$
$$\frac{Dv}{Dt} + fu = -\frac{1}{1+\sigma}\varphi_y$$
$$\frac{Dw}{Dt} = -\frac{1}{1+\sigma}\frac{1}{\overline{\rho}}(\varphi\overline{\rho})_z - g\frac{\sigma}{1+\sigma}$$

Expanding the first term on the r.h.s. of the z momentum equation allows us to rewrite the momentum equations as

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\frac{1}{1+\sigma}\nabla\varphi - \hat{\mathbf{k}}\frac{1}{1+\sigma}(\frac{1}{\overline{\rho}}\frac{\partial\overline{\rho}}{\partial z}\varphi + g\sigma)$$

The mass equation becomes

$$\overline{\rho}\sigma_t + \nabla\cdot(\overline{\rho}(1+\sigma)\mathbf{u}) = 0$$

or

$$\sigma_t + \nabla\cdot((1+\sigma)\mathbf{u}) + (1+\sigma)w\frac{1}{\overline{\rho}}\frac{\partial\overline{\rho}}{\partial z} = 0$$

The thermodynamic equation, divided by $\overline{\rho}$, becomes

$$\frac{D}{Dt}\sigma - \frac{1}{c_s^2}\frac{D}{Dt}\varphi + w[(1+\sigma)\frac{1}{\overline{\rho}}\frac{\partial\overline{\rho}}{\partial z} + \frac{g}{c_s^2} - \frac{\varphi}{c_s^2}\frac{1}{\overline{\rho}}\frac{\partial\overline{\rho}}{\partial z}] = \frac{\rho_T Q}{\overline{\rho} c_p}$$

upon substituting the definitions for the deviations in pressure and density.

Standard Boussinesq Form:

The usual form of the approximation is for a liquid. In the momentum equations it requires $\sigma << 1$ and $N^2 << gH$, where

$$N^2 = -g\frac{1}{\overline{\rho}}\frac{\partial\overline{\rho}}{\partial z} - \frac{g^2}{\overline{c}_s^2}$$

is the Brunt–Väisälä frequency. The momentum equations become

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\nabla\varphi - \hat{\mathbf{k}}g\sigma$$

and the mass equation simplifies to

$$\nabla \cdot \mathbf{u} = 0$$

In the thermodynamic equation, we must assume $\varphi << c_s^2\sigma$ (which is equivalent to $gH << c_s^2$ if the pressure and density deviations are related hydrostatically). We then find

$$\frac{D}{Dt}\sigma - w\frac{N^2}{g} = \frac{\overline{\rho}_T Q}{\overline{\rho} c_p}$$

(dropping the σ_T compared to $\overline{\rho}_T$ and ignoring the deviations in c_s^2 — all consistent with the assumptions above).

If we now define a buoyancy variable by

$$b = \int^z N^2 - g\sigma$$

and redefine the dynamic pressure

$$p^* = \varphi - \int^z\int^z N^2$$

we get the usual form of the Boussinesq model:

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\nabla p^* + b\hat{\mathbf{k}}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{D}{Dt}b = -g\frac{\overline{\rho}_T Q}{\overline{\rho} c_p}$$

You will often see $\alpha g T$ used in place of b in the above equations, where $\alpha \equiv -\rho_T/\rho$ is the thermal expansion coefficient.

Gas:

For a gas, the equations are a little different. They are much easier and more accurately derived in pressure coordinates, but it is possible to look at them in ordinary coordinates as well. In this case, it is advisable to treat the thermodynamic equation directly, using the perfect gas law:

$$\frac{D}{Dt}[\log\rho - \log p^{1/\gamma}] = \frac{\rho_T Q}{\rho c_p}$$

Substituting the expressions for the deviations gives

$$\frac{D}{Dt}\left[\log(1+\sigma) - \frac{1}{\gamma}\log(1+\frac{\overline{\rho}\varphi}{\overline{p}})\right] - w\frac{N^2}{g} = \frac{\rho_T Q}{\rho c_p}$$

Again $\sigma << 1$ and likewise φ is small, so that we can simplify to

$$\frac{D}{Dt}\left[\sigma - \frac{1}{\gamma}\frac{\overline{\rho}\varphi}{\overline{p}}\right] - w\frac{N^2}{g} = \frac{\overline{\rho}_T Q}{\overline{\rho} c_p}$$

suggesting a slightly different definition for buoyancy:

$$b = \int^z N^2 - g\sigma + \frac{g\overline{\rho}}{\gamma\overline{p}}\varphi$$

including somewhat more of the compressibility. The thermodynamic equation then is identical to that above. In the mass equation, however, we only drop the σ terms and find

$$\nabla\cdot\overline{\rho}\mathbf{u} = 0$$

The momentum equations are then simplified by dropping the σ terms and substituting the definition of buoyancy to find

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{k}}\times\mathbf{u} = -\nabla\varphi + b + \frac{N^2\varphi}{g}$$

The last term is then dropped (with less justification). Pressure coordinates allow incorporation of that term into the vertical derivatives.