

Laplace Tidal Equations — horizontal structure

Equations

After separation of variables of the original linearized, hydrostatic, pressure coordinate equations (with the “standard approximation” of neglecting the horizontal part of the rotation) are

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{u} + \mathbf{f} \times \mathbf{u} &= -grad \Phi \\ \frac{\partial}{\partial t}\Phi + gH_{eq} div \mathbf{u} &= 0\end{aligned}$$

Here \mathbf{f} is $f(\varphi)\hat{\mathbf{r}}$. The vertical structure of the horizontal velocities satisfies the equation

$$\frac{\partial}{\partial p} \frac{1}{S} \frac{\partial}{\partial p} F = -\frac{1}{gH_{eq}} F$$

with $S = -\Pi \bar{\theta}_p$. The solution to this, with suitable boundary conditions, defines the equivalent depths H_{eq} .

Solution of the horizontal eqns.

If we take $\mathbf{f} \times$ the momentum equations and subtract it from $\partial/\partial t$ of the same equations, we find

$$\frac{\partial^2}{\partial t^2}\mathbf{u} + f^2\mathbf{u} = -grad \frac{\partial}{\partial t}\Phi + \mathbf{f} \times grad \Phi$$

For motions with frequency ω , the velocity is given by

$$\mathbf{u} = -\frac{1}{f^2 - \omega^2} grad \frac{\partial}{\partial t}\Phi + \frac{\mathbf{f}}{f^2 - \omega^2} \times grad \Phi$$

Taking the divergence and substituting in the mass equation yields

$$\frac{\partial}{\partial t}\Phi - gH_{eq} div \frac{1}{f^2 - \omega^2} grad \frac{\partial}{\partial t}\Phi + gH_{eq} div \frac{\mathbf{f}}{f^2 - \omega^2} \times grad \Phi = 0$$

Using the vector identity

$$curl \Phi \mathbf{v} = grad \Phi \times \mathbf{v} + \Phi curl \mathbf{v}$$

gives

$$\frac{\partial}{\partial t}\Phi - gH_{eq} div \frac{1}{f^2 - \omega^2} grad \frac{\partial}{\partial t}\Phi + gH_{eq} div \left[\Phi curl \frac{\mathbf{f}}{f^2 - \omega^2} \right] = 0$$

For the spherical case, with $f = 2\Omega \sin \varphi$, the last term simplifies to

$$gH_{eq} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} \left[\Phi \frac{1}{a} \frac{\partial}{\partial \varphi} \frac{f}{f^2 - \omega^2} \right]$$

If we put in $\Phi \sim \exp(im\lambda - \omega t)$, we end up with a single equation for the meridional structure, with the frequency acting somewhat like the eigenvalue.

Small scale, high frequency

If we consider small scale waves in the sense that $grad \Phi/\Phi \gg grad f/f$, we can drop the last term and pull the $f^2 - \omega^2$ out to find

$$\left[1 - \frac{gH_{eq}}{f^2 - \omega^2} \nabla^2 \right] \Phi_t = 0$$

which has a zero frequency mode and a mode with

$$\omega^2 = f^2 + gH_{eq} \mathbf{k}^2$$

where the wavenumber is defined by the solutions of $\nabla^2 \Phi = -\mathbf{k}^2 \Phi$ as in the barotropic problem. These are the gravity waves.

Low frequency

The other mode has a low frequency (at least in the small scale limit) and satisfies

$$\frac{\partial}{\partial t} \Phi - gH_{eq} div \frac{1}{f^2} grad \frac{\partial}{\partial t} \Phi - gH_{eq} \frac{1}{a \cos \varphi} \frac{\beta}{f^2} \frac{\partial}{\partial \lambda} \Phi = 0$$

(at least away from the equator). In the case where the scales are still small, we can approximate this by

$$\frac{\partial}{\partial t} \left[\nabla^2 - \frac{f^2}{gH_{eq}} \right] \Phi + \beta \frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} \Phi = 0$$

a form of the Rossby wave equation. The local phase speed $c = a \cos \varphi (\omega/m)$ is given by

$$c = - \frac{\beta}{\mathbf{k}^2 + f^2/gH_{eq}}$$