13. The superposition principle

In the quasi-geostrophic system, the relationship between q_p anomalies and φ anomalies is linear, so that distributions of φ associated with individual anomalies of q_p can be superposed to form the full φ field associated with the full q_p distribution. The hydrostatic and geostrophic relations are also linear, so that perturbations of velocity and temperature also just superpose linearly. But *energy,* on the other hand, is a quadratic, and thus does not superpose linearly. This has important implications for energy transformations in quasi-balanced flows in general, and quasi-geostrophic flow, in particular. To see some of these implications, first form an energy integral for quasi-geostrophic flows. The equation for kinetic energy can be obtained by taking the dot product of the geostrophic flow vector, $\mathbf{V}_g$, with the quasi-geostrophic momentum equation (8.29) with the result

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla\right)\frac{1}{2}|\mathbf{V}_g|^2 = -f\mathbf{V}_g \cdot \hat{k} \times \mathbf{V} + \mathbf{V}_g \cdot \mathbf{F}, \tag{13.1}$$

where we have made use of the vector identity

$$\mathbf{A} \cdot \hat{k} \times \mathbf{A} = 0,$$

for any vector $\mathbf{A}$. By using the geostrophic relation (8.32) in (13.1), the latter may be written

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla\right)\frac{1}{2}|\mathbf{V}_g|^2 = -\mathbf{V} \cdot \nabla\varphi + \mathbf{F} \cdot \mathbf{V}_g$$

$$= -\nabla \cdot (\mathbf{V}\varphi) - \varphi\frac{\partial \omega}{\partial p} + \mathbf{F} \cdot \mathbf{V}_g, \tag{13.2}$$

where we have made use of the mass continuity equation, (8.31). This is the quasi-geostrophic kinetic energy equation.

An equation for potential and internal energy can be formed by multiplying (9.4) by $\partial\varphi/\partial p$:

$$\left(\frac{\partial}{\partial t}+\mathbf{V}_g\cdot\nabla\right)\frac{1}{2\mathcal{S}}\left(\frac{\partial\varphi}{\partial p}\right)^2+\omega\frac{\partial\varphi}{\partial p}=-\frac{\partial\varphi}{\partial p}\overline{\alpha}\frac{\dot{Q}}{\mathcal{S}\overline{\theta}}. \tag{13.3}$$

An equation for total energy associated with perturbations can be formed by summing (13.2) and (13.3):

$$\left(\frac{\partial}{\partial t}+\mathbf{V}_g\cdot\nabla\right)\left[\frac{1}{2}|\mathbf{V}_g|^2+\frac{1}{2}\frac{1}{\mathcal{S}}\left(\frac{\partial\varphi}{\partial p}\right)^2\right]=-\nabla\cdot(\mathbf{V}\varphi)-\frac{\partial}{\partial p}(\omega\varphi)+\mathbf{F}\cdot\mathbf{V}_g-\frac{\partial\varphi}{\partial p}\overline{\alpha}\frac{\dot{Q}}{\mathcal{S}\overline{\theta}}.$$
$$\tag{13.4}$$

Energy is locally changed by a divergence of the flux (by the actual wind) of perturbation geopotential, by heating and by friction.

We next integrate (13.4) over a three-dimensional volume defined in such a way that there is no net flux of geopotential or energy itself across its boundaries. This integration (after applying the divergence theorem) then results in

$$\int_v\frac{\partial E_p}{\partial t}dV=\int_v\left(\mathbf{F}\cdot\mathbf{V}_g-\frac{\partial\varphi}{\partial p}\overline{\alpha}\frac{\dot{Q}}{\mathcal{S}\overline{\theta}}\right), \tag{13.5}$$

where E_p is the pseudo energy, defined

$$E_p\equiv\frac{1}{2}|\mathbf{V}_g|^2+\frac{1}{2\mathcal{S}}\left(\frac{\partial\varphi}{\partial p}\right)^2. \tag{13.6}$$

This shows that the integral of E_p over a suitably defined volume is conserved in the absence of heating and friction.

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We can now show that E_p is related to an integral of the pseudo potential vorticity. Begin with the perturbation pseudo potential vorticity defined by (9.14), multiply it by $-\varphi$ and integrate the result over a control volume on whose lateral sides either φ or its normal gradient vanishes:

$$-\int_v \varphi q_p' dV = -\frac{1}{f_0} \int_v \varphi \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \right) dV - f_0 \int_v \varphi \frac{\partial}{\partial p} \left(\frac{1}{\mathcal{S}} \frac{\partial \varphi}{\partial p} \right) dV. \qquad (13.7)$$

We next integrate the two terms on the right side of (13.7) by parts and use the geostrophic relations to get

$$-\int_v \varphi q_p' dV = f_0 \int_v \left(|\mathbf{V}_g|^2 + \frac{1}{\mathcal{S}} \left(\frac{\partial \varphi}{\partial p} \right)^2 \right) dV$$

$$-\frac{1}{f_0} \int_v \left[\frac{\partial}{\partial x} \left(\varphi \frac{\partial \varphi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\varphi \frac{\partial \varphi}{\partial y} \right) \right] dV - f_0 \int_A \frac{1}{\mathcal{S}} \varphi \frac{\partial \varphi}{\partial p} \Big|_{p_t}^{p_0} dA, \qquad (13.8)$$

where the last integral is over the horizontal areas bounding the top and bottom of the control volume.

By our assumption that either φ or its normal gradient vanishes at the lateral boundaries of the volume, the second term on the right side of (13.8) vanishes, and combining the last term on the right with the left side of (13.8) gives

$$-\int_v \varphi \left[q_p' - \frac{f_0}{\mathcal{S}} \frac{\partial \varphi}{\partial p} \delta(p_0 - p) + \frac{f_0}{\mathcal{S}} \frac{\partial \varphi}{\partial p} \delta(p - p_t) \right] = 2 f_0 \int_v E_p \, dV, \qquad (13.9)$$

where we have substituted (13.6).

Note that using the arguments presented in section 11, the left side of (13.9) is simply the integral of the product of φ with the pseudo potential vorticity, *including*

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the effective delta functions at horizontal boundaries that result when there are temperature perturbations there. This integral is proportional to the pseudo energy of the system.

A general conclusion that can be reached with the aid of (13.9) is that *bringing together like-signed potential vorticity anomalies (or their equivalent delta functions in the form of boundary θ' anomalies) entails an increase in the energy associated with the anomalies.* An example will suffice to show why this follows. Suppose we have two delta function q_p anomalies in an infinite domain, separated by a great distance, as in the top of Figure 13.1. Let us suppose that each delta function has an amplitude of 1 in some suitably normalized three-dimensional coordinates.

Now the inversion of the elliptic relationship between q_p' and φ will result in a field of φ that decays away from each of the two point potential vortices. Let us suppose that φ has been normalized in such a way that its value at the location of the point potential vortex is -1. Let us also suppose that the two vortices are so far apart that, for all practical purposes, the amplitude of the part of φ associated with one point potential vortex is zero at the location of the other point potential vortex. In that case, the integral at the left of (13.9) will have the value of 4π associated with each vortex, or 8π total.

Suppose that these two point potential vortices are brought together by some process and combined into a single point potential vortex, as at the bottom of Figure 13.1. Now there are 2 units of q_p' in the combined vortex, and the amplitude of φ
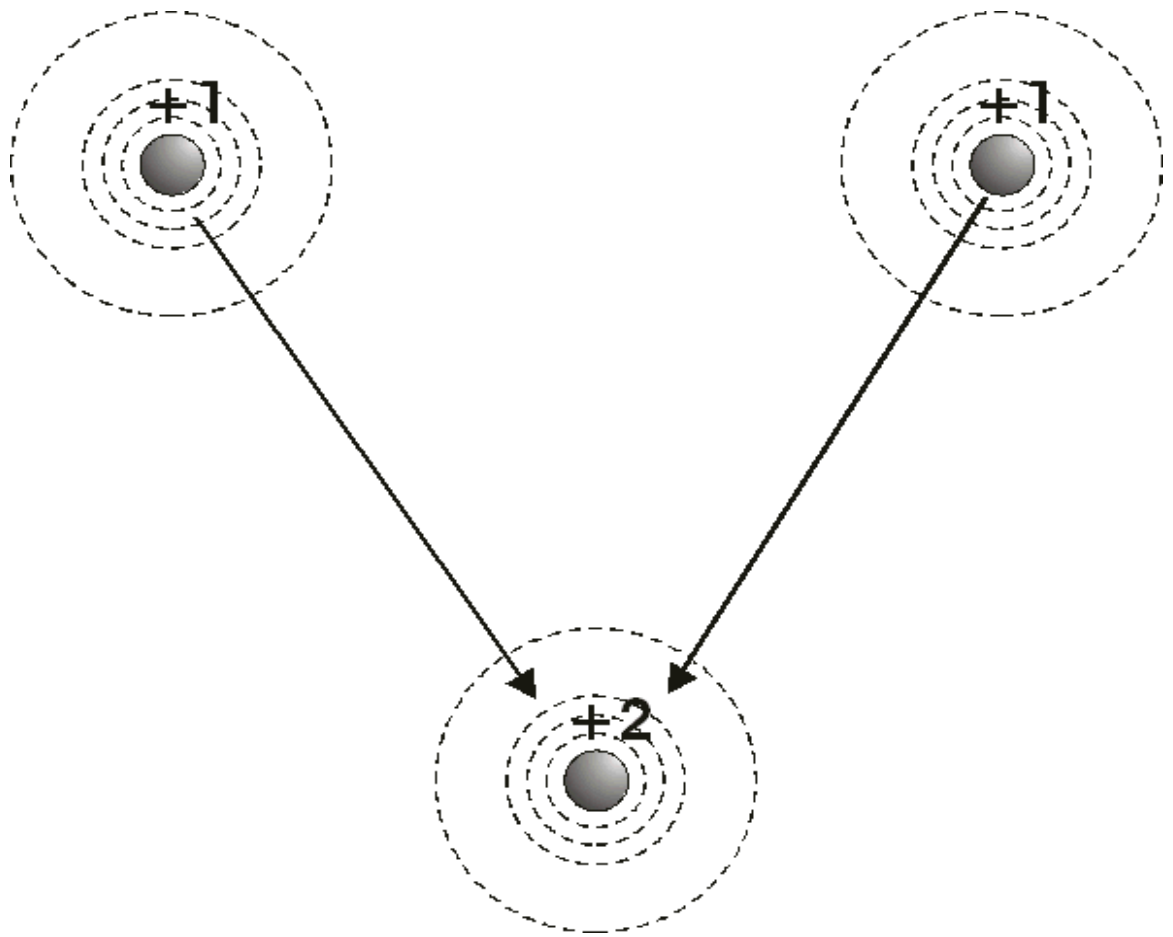
Figure 13.1

at the location of the combined vortex is thus -2. This gives 16π energy units, according to (13.9)—double what was there before!

Thus it takes a source of energy (from, say, the background flow) to rearrange potential vorticity into more compact masses, for which the energy anomaly is greater. For a given mass of potential vorticity, the maximum energy is achieved when the potential vorticity is concentrated in a sphere, in a coordinate system scaled by the deformation radius. The minimum energy content occurs when the

potential vorticity is distributed in an infinitely long thread of zero thickness.

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