

## 19. Baroclinic Instability

In two-dimensional barotropic flow, there is an exact relationship between mass streamfunction  $\psi$  and the conserved quantity, vorticity ( $\eta$ ) given by  $\eta = \nabla^2\psi$ . The evolution of the conserved variable  $\eta$  in turn depends only on the spatial distribution of  $\eta$  and on the flow, which is derivable from  $\psi$  and thus, by inverting the elliptic relation, from  $\eta$  itself. This strongly constrains the flow evolution and allows one to think about the flow by following  $\eta$  around and inverting its distribution to get the flow.

In three-dimensional flow, the vorticity is a vector and is not in general conserved. The appropriate conserved variable is the potential vorticity, but this is not in general invertible to find the flow, unless other constraints are provided. One such constraint is geostrophy, and a simple starting point is the set of quasi-geostrophic equations which yield the conserved and invertible quantity  $q_p$ , the pseudo-potential vorticity.

The same dynamical processes that yield stable and unstable Rossby waves in two-dimensional flow are responsible for waves and instability in three-dimensional baroclinic flow, though unlike the barotropic 2-D case, the three-dimensional dynamics depends on at least an approximate balance between the mass and flow fields.

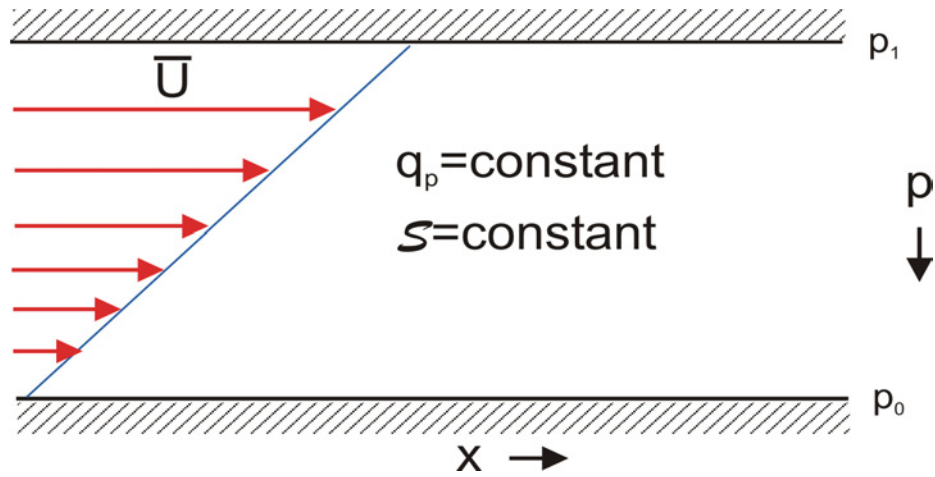


Figure 19.1

a. *The Eady model*

Perhaps the simplest example of an instability arising from the interaction of Rossby waves in a baroclinic flow is provided by the *Eady Model*, named after the British mathematician Eric Eady, who published his results in 1949. The equilibrium flow in Eady's idealization is illustrated in Figure 19.1. A zonal flow whose velocity increases with altitude is confined between two rigid, horizontal plates. This flow is in exact thermal wind balance with an equatorward-directed temperature gradient and is considered to have constant pseudo potential vorticity,  $q_p$ , as well as constant background static stability,  $\mathcal{S}$ . The flow occurs on an  $f$  plane, so  $\beta = 0$ . Evolution of the flow is taken to be inviscid and adiabatic.

At first blush, it might appear that no interesting quasi-geostrophic dynamics can occur in this system since there are no spatial gradients of  $q_p$  and thus no Rossby waves. But there are temperature gradients on both boundaries, and according to

the analysis presented in Sections 11 and 12, Eady edge waves—boundary-trapped Rossby waves—can exist. Eady showed that the two sets of Rossby waves corresponding to both boundaries can interact unstably, giving rise to exponential instability.

Since pseudo potential vorticity is conserved in this problem, and since it is initially constant, perturbations to it vanish. According to (9.11),

$$\frac{1}{f_0} \nabla^2 \varphi + \frac{f_0}{\mathcal{S}} \frac{\partial^2 \varphi}{\delta p^2} = 0, \quad (19.1)$$

which was already derived as (12.1). Here  $\varphi$  is now defined as a perturbation to the background geopotential distribution.

The background zonal flow is specifically defined to be linear in pressure:

$$\frac{d\bar{u}}{dp} = \frac{R}{f_0 p_0} \frac{d\bar{\theta}}{dy} = -\gamma, \quad (19.2)$$

where  $\bar{u}$  is the background zonal wind,  $\bar{\theta}$  is the background potential temperature,  $f_0$  is the Coriolis parameter and  $R$  is the gas constant. We introduce  $\gamma$  for notational convenience. For the ocean,  $\gamma$  would be defined as  $-G\bar{\sigma}_y$ , evaluated at a suitable pressure level.

Integrating (19.2) in pressure, we get a relationship between the background zonal velocities at the two boundaries:

$$\bar{u}_1 - \bar{u}_0 = \gamma(p_0 - p_1), \quad (19.3)$$

where  $p_0$  and  $p_1$  are the pressures at the two boundaries. The system is Galilean invariant, so we can add an arbitrary constant to the background zonal flow. We

choose this so that the background zonal flow at one boundary is equal in magnitude but opposite in sign to the flow at the other boundary, to wit

$$\begin{aligned} u_1 &= \frac{1}{2}\gamma(p_0 - p_1) \equiv \frac{1}{2}\Delta u, \\ u_0 &= -\frac{1}{2}\gamma(p_0 - p_1) \equiv -\frac{1}{2}\Delta u, \end{aligned} \tag{19.4}$$

where  $\Delta u$  is the background shear,  $\bar{u}_1 - \bar{u}_0$ . From (19.3) we have

$$\Delta u = \gamma \Delta p, \tag{19.5}$$

where  $\Delta p \equiv p_0 - p_1$ .

To solve (19.1), we need to impose boundary conditions. We take perturbations to the background to be periodic in the two horizontal directions. In the vertical, the appropriate boundary conditions are given by (11.1):

$$\left( \frac{\partial}{\partial t} + \mathbf{V}_g \cdot \nabla \right) \theta = 0 \text{ on } p = p_0, p_1. \tag{19.6}$$

As in section 12, we *linearize* this boundary condition around the background zonal flow, assuming that perturbations to it are so small that contributions to (19.6) that are quadratic in the perturbations can be neglected. (Note that (19.6) is the *only* equation in Eady's system that is linearized.) Linearization of (19.6) gives

$$\left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \theta' + v' \frac{d\bar{\theta}}{dy} = 0 \text{ on } p = p_0, p_1. \tag{19.7}$$

Using the hydrostatic equation for  $\theta'$ , the geostrophic relation for  $v'$  and (19.2) for  $d\bar{\theta}/dy$  gives

$$\left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial \varphi}{\partial p} + \gamma \frac{\partial \varphi}{\partial x} = 0 \text{ on } p = p_0, p_1. \tag{19.8}$$

Specializing this to the upper and lower boundaries of the Eady model using (19.4)

gives

$$\begin{aligned} \left( \frac{\partial}{\partial t} + \frac{1}{2} \Delta u \frac{\partial}{\partial x} \right) \frac{\partial \varphi}{\partial p} + \gamma \frac{\partial \varphi}{\partial x} &= 0 \text{ on } p = p_1, \\ \left( \frac{\partial}{\partial t} - \frac{1}{2} \Delta u \frac{\partial}{\partial x} \right) \frac{\partial \varphi}{\partial p} + \gamma \frac{\partial \varphi}{\partial x} &= 0 \text{ on } p = p_0. \end{aligned} \quad (19.9)$$

Thus the mathematical problem to be solved is given by (19.1) coupled to (19.9), remembering that we are applying periodic boundary conditions in  $x$  and  $y$ .

Since (19.1) is a linear elliptic equation with constant coefficients, we look for solutions in terms of exponential normal modes of the form

$$\varphi = [A \sinh(rp) + B \cosh(rp)] e^{ik(x-ct)+ily}, \quad (19.10)$$

where  $c$  is a (potentially complex) phase speed, and  $r$ ,  $k$ , and  $l$  are wavenumbers in pressure and in  $x$  and  $y$ , respectively. From (19.1) we have

$$r^2 = \frac{\mathcal{S}}{f_0^2} (k^2 + l^2), \quad (19.11)$$

which shows that the vertical exponential decay scale of the disturbances is related to a measure of the horizontal scale by the Rossby aspect ratio  $\frac{f_0}{\sqrt{\mathcal{S}}}$ .

It proves convenient to nondimensionalize the complex phase speed,  $c$ , and the vertical wavenumber,  $r$ , according to

$$c \rightarrow \Delta u c, \quad (19.12)$$

$$r \rightarrow r / \Delta p.$$

Making use of these and substituting (19.10) into the two vertical boundary conditions (19.9) gives the dispersion relation:

$$c^2 = \frac{1}{4} + \frac{1}{r^2} - \frac{\text{ctnh}(r)}{r}. \quad (19.13)$$

At first take, it would appear that the time dependence of the normal modes of the Eady problem is independent of the particular values of the horizontal wavenumbers  $k$  and  $l$ , depending only on their combination  $k^2 + l^2$  through (19.11). But if  $c$  has a nonzero imaginary part,  $c_i$ , then (19.10) shows that the exponential growth rate is given by  $kc_i$ , so we shall be concerned about  $k$  as well.

Although (19.13) can be easily graphed, it is interesting to explore certain limiting and special cases. In the small horizontal wavelength limit, we have

$$\lim_{r \rightarrow \infty} c^2 = \frac{1}{4} + \frac{1}{r^2} - \frac{1}{r} = \left( \frac{1}{2} - \frac{1}{r} \right)^2, \quad (19.14)$$

whose solution is

$$c = \pm \left( \frac{1}{2} - \frac{1}{r} \right). \quad (19.15)$$

These are just the solutions of the Eady edge wave problem solved in section 12, in the limit of large  $r$ , with the positive root corresponding to the upper boundary. In nondimensional terms,  $\frac{1}{2}$  corresponds to the background flow at the upper boundary, while  $-\frac{1}{2}$  corresponds to the background flow at the lower boundary. So these are small, stable Eady waves at each boundary, swimming upstream. This is the same as the asymptotic solution of the Eady edge wave at each boundary independently, given by (12.13), so in this limit, the two edge waves pass each other like ships in the night, ignorant of each other's existence.

In the limit of large wavelength (small  $r$ ), meaningful solutions require us to expand the  $ctnh(r)$  term in (19.13) to second order, to wit

$$\lim_{r \rightarrow 0} rctnh(r) = 1 + \frac{1}{2}r^2.$$

This gives

$$\lim_{r \rightarrow 0} c^2 = -\frac{1}{4},$$

or

$$\lim_{r \rightarrow 0} c = \pm \frac{1}{2}i. \quad (19.16)$$

Substitution into (19.10) shows that these modes have vanishing phase speed but grow or decay at an exponential rate given by

$$\sigma \equiv kc_i = \pm \frac{1}{2}k. \quad (19.17)$$

Thus longwave modes of the Eady model are stationary and grow or decay exponentially in time.

Examination of the dispersion relation (19.13) shows that  $c^2 = 0$  for a particular value of  $r$  which turns out to be  $\simeq 2.4$ . Also, in the exponential regime at long wavelength, the quantity  $c^2 r^2$  has an extremum when  $r = 1.606$  corresponding to a value of  $rc_i$  of 0.3098. From (19.11), we have that

$$k = \sqrt{\frac{r_0^2}{S} r^2 - l^2},$$

so the maximum growth rate,  $kc_i$ , is given by

$$(kc_i)_{\max} = 0.3098 \sqrt{1 - \frac{l^2}{r^2}},$$

where we have used a suitable nondimensionalization of  $l$ . This shows that *the maximum growth rate always occurs for  $l = 0$* , i.e., for disturbances that are independent of  $y$ .

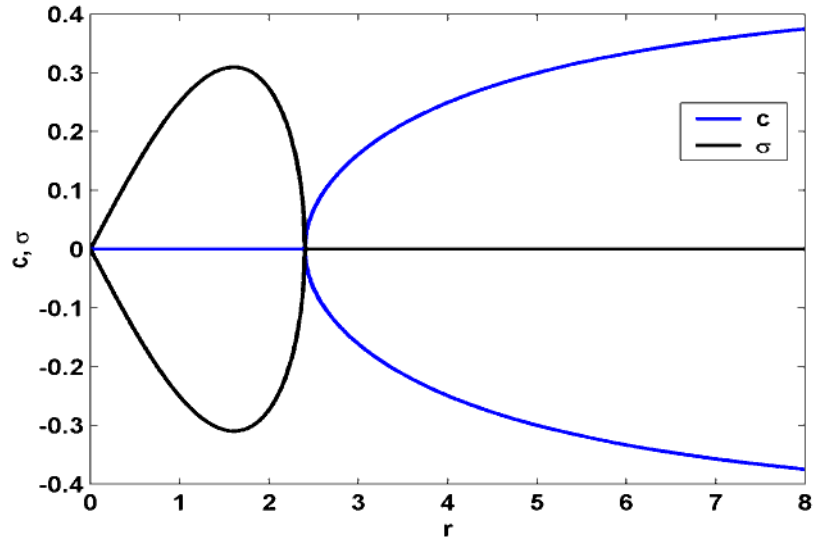


Figure 19.2

The complete solution to (19.13) is graphed in Figure 19.2. For  $r < 2.4$ , the modes are nonpropagating and exponentially growing or decay. For  $r > 2.4$ , there are two neutral propagating roots corresponding to eigenfunctions that maximize at one or the other boundary—the two Eady edge waves.

Note that the solutions to the Eady problem in Figure 19.2 closely resemble the solutions of the Rayleigh barotropic instability problem discussed in section 18.1 and shown in Figure 18.4. In fact, *the dynamics are essentially the same*; the only difference is one of geometry: whereas the Rossby waves in the barotropic Rayleigh problem interact laterally, those in the baroclinic Eady problem interact vertically. But *there is no fundamental difference between barotropic and baroclinic instability*, although in the pure barotropic case the disturbance energy is drawn from the kinetic energy of the mean flow, whereas in the pure baroclinic case it is



drawn from the potential energy inherent in the background horizontal temperature gradient.

As in Rayleigh's problem, the instability of the Eady basic state can be usefully regarded as resulting from the mutual amplification of phase-locked Rossby waves, as illustrated in Figure 19.3. If a cold anomaly at the upper boundary is positioned west of a warm anomaly at the lower boundary, invertibility gives cyclonic circulation at the location of each of the two boundary temperature anomalies, decaying exponentially away from the boundary. The cyclonic circulation associated with the upper cold anomaly, projected down the lower boundary gives a poleward flow at the location of the lower warm anomaly. Advection of the background temperature gradient leads to a positive temperature tendency there, reinforcing the existing lower boundary temperature anomaly. Likewise, the cyclonic circulation associated with the lower warm anomaly, projecting up to the upper boundary, causes a temperature advection that amplifies the upper cold anomaly. Note, however, that there are small phase shifts between the boundary temperature anomalies and the temperature advection, owing to the circulations induced by the temperature anomalies at the opposite boundaries. These phase shifts serve to alter the propagation speeds of the disturbances, keeping them phase-locked.

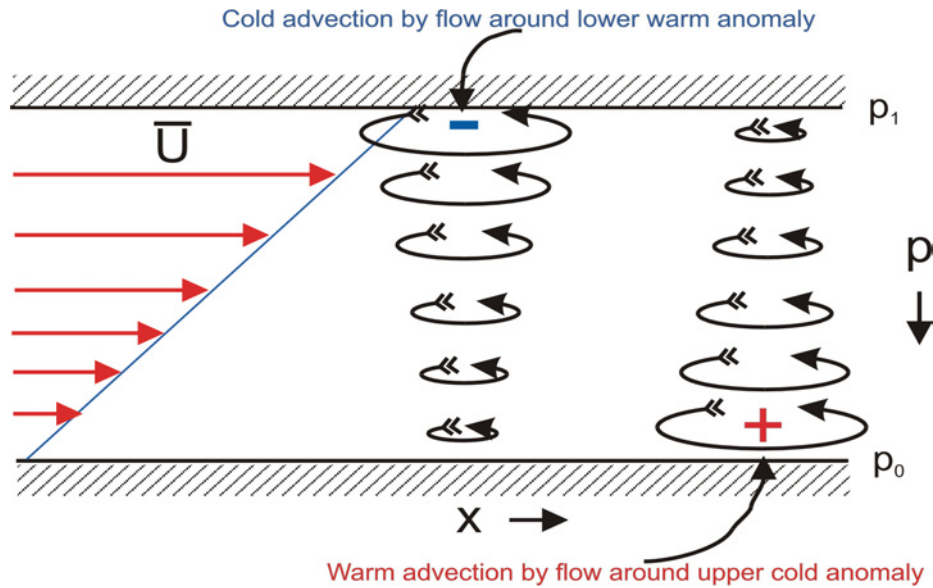


Figure 19.3

*b. The Charney Model*

At the same time Eady was developing his model of baroclinic instability, Jule Charney, then a graduate student at UCLA, was working on a somewhat different model, which he ultimately published in 1947. Charney used essentially the quasi-geostrophic equations, and took as his basic state one of a zonal wind increasing linearly with altitude, as in the Eady model. But unlike the latter, Charney did not apply an upper lid, allowing his domain to be semi-infinite, and instead of having constant pseudo-potential vorticity, he took the background state to have a constant meridional gradient of  $q_p$ . Following Charney's original paper, we here work in height coordinates, rather than pressure coordinates. One can show that

in height coordinates,  $q_p$  is given by

$$q_p = \frac{1}{f_0} \nabla^2 \frac{p}{\rho_0} + \beta y + \frac{f_0}{\bar{\rho}} \frac{\partial}{\partial z} \frac{\bar{\rho}}{\rho_0 N^2} \frac{\partial p}{\partial z}, \quad (19.18)$$

where  $p$  is the perturbation of pressure away from the background state,  $\rho_0$  is a constant reference density,  $\bar{\rho}(z)$  is the density distribution of the background state, and

$$N^2 \equiv \frac{g}{\theta_0} \frac{d\bar{\theta}}{dz}, \quad (19.19)$$

where  $\bar{\theta}(z)$  is the potential temperature of the background state. Note that  $N$  has the units of inverse time and is called the buoyancy frequency, or the Brunt-Vaisälä frequency.

Charney took  $N^2 = \text{constant}$  and

$$\bar{\rho} = \rho_0 e^{-z/H}, \quad (19.20)$$

with  $H$  a (constant) density scale height. Then (19.18) becomes

$$q_p = \frac{1}{f_0} \nabla^2 \frac{p}{\rho_0} + \beta y + \frac{f_0}{N^2 \rho_0} \frac{\partial^2 p}{\partial z^2} - \frac{f_0}{HN^2 \rho_0} \frac{\partial p}{\partial z}. \quad (19.21)$$

As mentioned before, Charney took his basic state  $q_p$  to have a constant meridional gradient:

$$\frac{d\bar{q}_p}{dy} = \frac{1}{f_0} \nabla^2 \frac{1}{\rho_0} \frac{d\bar{p}}{dy} + \beta - \frac{f_0}{N^2 \rho_0} \frac{\partial^2}{\partial z^2} \frac{d\bar{p}}{dy} - \frac{f_0}{HN^2 \rho_0} \frac{\partial}{\partial z} \frac{d\bar{p}}{dy}. \quad (19.22)$$

But, using the geostrophic relation

$$\frac{1}{\rho_0} \frac{d\bar{p}}{dy} = -f_0 \bar{u}$$

and remembering that Charney took  $\bar{u}$  to be a linear function of  $z$ ,

$$\frac{d\bar{q}_p}{dy} = \beta + \frac{f_0^2}{HN^2} \frac{d\bar{u}}{dz} = \text{constant} \equiv \hat{\beta}. \quad (19.23)$$

Here we use  $\hat{\beta}$  to denote the constant background pseudo-potential vorticity gradient. We will also define

$$\Lambda \equiv \frac{d\bar{u}}{dz}$$

and take

$$\bar{u} = \Lambda z. \quad (19.24)$$

Linearizing the pseudo-potential vorticity equation (9.10) about this state gives

$$\left( \frac{\partial}{\partial t} + \Lambda z \right) q'_p + v' \hat{\beta} = 0, \quad (19.25)$$

where  $q'$  is the perturbation pseudo-potential vorticity, which from (19.21) is given by

$$q'_p = \frac{1}{f_0} \nabla^2 \frac{p'}{\rho_0} + \frac{f_0}{N^2 \rho_0} \frac{\partial^2 p'}{\partial z^2} - \frac{f_0}{HN^2 \rho_0} \frac{\partial p'}{\partial z}. \quad (19.26)$$

Charney's lower boundary condition is identical to Eady's, given that the constant vertical shear of the background zonal wind must be associated with a constant background meridional gradient of potential temperature. Making explicit use of the hydrostatic and thermal wind equation, we have as a lower boundary condition

$$\frac{\partial}{\partial t} \frac{\partial p'}{\partial z} - \Lambda \frac{\partial p'}{\partial x} = 0 \text{ on } z = 0. \quad (19.27)$$

Charney applied a *wave radiation condition* at  $z \rightarrow \infty$ . This asserts that, away from the origin of the waves, the wave energy propagation must be away from the source.

In this case, it implies that wave energy must be travelling upward through the top of the domain. On the other hand, Charney was primarily interested in growing (unstable) disturbances. Such waves should decay exponentially away from their source, so

$$\lim_{z \rightarrow \infty} p' = 0, \quad (19.28)$$

which we apply as an upper boundary condition.

Since the coefficients of (19.26), (19.27), and (19.28) are constant in  $x$ ,  $y$ , and time, we can look for normal mode solutions of the form

$$p' = \hat{p}(z)e^{ik(x-ct)+ily}, \quad (19.29)$$

where  $c$  is complex.

Substituting into (19.26) and the boundary conditions (19.27) and (19.28) gives

$$\frac{d^2 \hat{p}}{dz^2} - \frac{1}{H} \frac{d\hat{p}}{dz} + \frac{N^2}{f_0^2} \left[ \frac{\beta^2}{\Lambda z - c} - k^2 - l^2 \right] \hat{p} = 0, \quad (19.30)$$

$$c \frac{d\hat{p}}{dz} + \Lambda \hat{p} = 0 \text{ on } z = 0, \quad (19.31)$$

and

$$\hat{p} = 0 \text{ on } z = \infty \text{ (for } c_i > 0 \text{)}. \quad (19.32)$$

Since (19.3) has a nonconstant coefficient, its solution is not in terms of simple trigonometric functions. Nevertheless, it can be put in a canonical form by a suitable substitution of variables, and solutions can be obtained in terms of confluent hypergeometric functions.

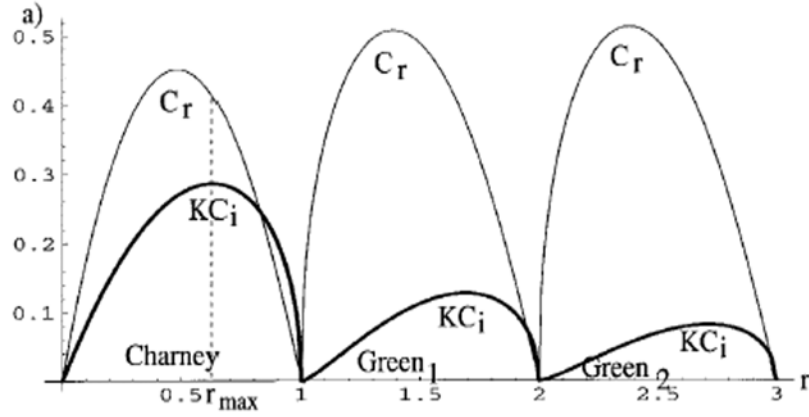


Figure 19.4

An example of solutions for the complex phase speed  $c$  is shown in Figure 19.4, while an example of eigenfunctions of unstable modes is shown in Figure 19.5.

Once again, the most unstable modes do not vary with  $y(l = 0)$ . For the Boussinesq limit ( $H \rightarrow \infty$ ), the maximum growth rate is given by

$$\sigma_{\max} = 0.286 \frac{f_0}{N} \Lambda, \quad (19.33)$$

which may be compared to the maximum growth rate in the Eady model of  $0.31 \frac{f_0}{N} \Lambda$ .

This maximum occurs at a horizontal wavenumber  $k_{\max}$  whose inverse is given by

$$k_{\max}^{-1} = 1.26 \frac{f_0}{\hat{\beta} N} \Lambda. \quad (19.34)$$

Note that the *maximum* growth rate (19.33) is independent of  $\hat{\beta}$ , but the wavelength of maximum growth is proportional to  $\hat{\beta}^{-1}$ .

In the Charney model, the surface Eady edge wave, propagating eastward, interacts unstably with an *internal* Rossby wave, living on the background  $q_p$  gradient and travelling westward relative to the flow (as opposed to another Eady

edge wave, as in the Eady model). As we have seen repeatedly, when looking at the Rayleigh instability problem (section 18) or the Eady problem (section 19a), counter-propagating Rossby waves *must* phase lock for exponential instability. This requirement determines the horizontal and vertical scales of unstable modes in the Charney problem, whereas in the Eady model they are determined by the imposed depth of the system,  $H$ .

We can derive the parametric dependence of the wavelength of maximum instability given by (19.34) from the requirement of phase-locking as follows.

First, the Eady edge wave propagates eastward at a rate given approximately by (12.12) and (12.13) which, specialized to the present problem with  $l = 0$ , is

$$c_{\text{Eady}} \simeq \frac{f_0 \Lambda}{N} \frac{1}{k}. \quad (19.35)$$

Remember that this is just the background zonal wind speed at the altitude of the Rossby penetration depth,  $f_0/NK$ . On the other hand, the ground-relative phase speed of a free internal Rossby wave of zonal wavenumber  $k$  is

$$c_{\text{Rossby}} \simeq \tilde{u} - \frac{\hat{\beta}}{k^2}, \quad (19.36)$$

where  $\tilde{u}$  is some average background wind in the layer containing the Rossby wave.

We assume that  $\tilde{u}$  scales with the mean wind at the Rossby penetration depth:

$$\tilde{u} \simeq \mu \Lambda \frac{f_0}{Nk},$$

where  $\mu$  is some number, presumably less than unity, so

$$c_{\text{Rossby}} \simeq \mu \Lambda \frac{f_0}{Nk} - \frac{\hat{\beta}}{k^2}. \quad (19.37)$$

For phase locking, we equate  $c_{\text{Eady}}$ , given by (19.35), to  $c_{\text{Rossby}}$ , given by (19.37) to get

$$k^{-1} \simeq (1 - \mu) \frac{f_0}{N} \frac{\Lambda}{\hat{\beta}}, \quad (19.38)$$

which is the same scale as (19.34). Thus the most unstable wave has horizontal (and therefore vertical) dimensions that allow it to interact optimally with the surface Eady edge wave.

The vertical scale of the most unstable mode is just the Rossby penetration depth based on the horizontal scale given by (19.34):

$$h \simeq \frac{f_0^2}{N^2} \frac{\Lambda}{\hat{\beta}}.$$

For typical atmospheric values of  $f_0$ ,  $N$ ,  $\Lambda$ , and  $\hat{\beta}$ , this is of order 10 km—curiously close to the actual height of the tropopause.



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