Two layer adjustment

(See Dewar and Killworth(1990), *J. Phys. Oceanogr.*, **20**, 1563-75 for a more accurate treatment.)

If the upper layer has density ρ_1 and thickness h_1, and the lower layer has ρ_2, h_2, the upper layer pressure is

$$\frac{1}{\rho_1}\nabla p_1 = g\nabla(h_1 + h_2)$$

while the lower layer pressure is

$$\frac{1}{\rho_2}\nabla p_2 = \frac{\rho_1}{\rho_2} g\nabla h_1 + g\nabla h_2$$

From geostrophic balance for the azimuthal flow (neglecting the cyclostrophic terms), we get

$$f(v_1 - v_2) = g'\frac{\partial}{\partial r}h_1 \quad , \quad g' = \frac{\rho_2 - \rho_1}{\rho_2}g$$

Conservation of PV tells us

$$\zeta_i + f = f\frac{h_i(r)}{h_i^0(r_{0,i})}$$

where $r_{0,i}$ is the initial radius of the annulus which settles at radius r and h_i^0 represents the initial thickness at the initial position.

If we subtract the two PV equations, we find

$$\zeta_1 - \zeta_2 = \frac{g'}{f}\nabla^2 h_1 = f\frac{h_1(r)}{h_1^0(r_{0,1})} - f\frac{h_2(r)}{h_2^0(r_{0,2})}$$

Since we've already neglected order Rossby number terms in the balance statement, we might as well linearize the h values also:

$$h_1 = H_1 + \eta - h \quad , \quad h_2 = H_2 + h$$

where η is the surface displacement and h the interface displacement. Our equation becomes

$$\frac{g'}{f}\nabla^2(\eta - h) = \frac{f}{H_1}(\eta - \eta^0 - h + h^0) - \frac{f}{H_2}(h - h^0)$$

and we also ignore the difference between r_0 and r. Finally, we neglect surface displacements compared to interface displacements and find

$$\nabla^2 h = R_d^{-2}(h - h^0)$$

with $R_d^{-2} = f^2/g'H_1 + f^2/g'H_2$.

Exterior: Outside $r = a$, we have $h_0 = 0$ and

$$h = AK_0(r/R_d)$$

Interior: Inside $r = a$, the initial PV differs from the value at infinity so that $h_0 = H$ and

$$h = H + BI_0(r/R_d)$$

Matching together the values and the slopes of h gives

$$H + BI_0(a/R_d) = AK_0(r/R_d) \quad \textit{and} \qquad BI_1(a/R_d) = -AK_1(a/R_d)$$

Our solution is therefore

$$h = \begin{cases} H[1 - \frac{a}{R_d}K_1(\frac{a}{R_d})I_0(\frac{r}{R_d})] & r < a \\ \\ HI_1(\frac{a}{R_d})K_0(\frac{r}{R_d}) & r > a \end{cases}$$

Radial profiles for different $\gamma = a/R_d$ values ($\gamma = 0.25$ is the lowest curve).

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