Chapter 3

Transport of passive and active tracers in turbulent flows

A property of turbulence is to greatly enhance transport of tracers. For example, a dissolved sugar molecule takes years to diffuse across a coffee cup, relying only on molecular agitation (actually on that time scale the coffee will surely evaporate). With a spoon, the coffee drinker can create eddies that transport dissolved sugar throughout the cup in less than a second. This enhanced transport is generally described as an *eddy diffusivity*.

The concept of eddy diffusivity is often justified by appealing to an analogy between turbulent eddies and molecular diffusion. The argument goes that turbulent eddies move tracer parcels in erratic motions, much alike bombardment by molecular agitation in Brownian motion, and thus the action of turbulence may be represented as an enhanced diffusion. This concept was formalized by Taylor's theory of turbulent dispersion. Taylor showed that the spreading of particles in a turbulent field can be described in terms of an eddy diffusivity. The diffusivity introduced by Taylor is a Lagrangian quantity defined in terms of the Lagrangian velocity experienced by particles. In this lecture we show that, under certain approximation to be discussed, the concept of eddy diffusivity can be extended to Eulerian variables. The Eulerian eddy diffusivity describes the spreading of tracers rather than particles. Can you guess what is the difference?

3.1 Tracer transport in Eulerian coordinates

We approach the parameterization of eddy transport of tracers in a systematic way. We derive an equation for the mean tracer concentration where the eddy effects are represented as the sum of a tracer diffusion by an effective diffusivity and advection by a Stokes drift. We will compare this result with the Lagrangian description of dispersion.

Once again we use a Reynolds decomposition of variables into mean and eddy components. We depart from the advection-diffusion equation for a generic passive tracer of concentration c,

$$c_t + \boldsymbol{u} \cdot \nabla c = \kappa \nabla^2 c, \tag{3.1}$$

where κ is the molecular diffusivity and $\boldsymbol{u}$ is an incompressible ($\nabla \cdot \boldsymbol{u} = 0$) velocity field. The velocity field is given in this problem, i.e. we do not write a momentum equation to solve for the velocity field. The equation for the mean tracer concentration is,

$$\bar{c}_t + \bar{\boldsymbol{u}} \cdot \nabla \bar{c} + \nabla \cdot \overline{\boldsymbol{u}' c'} = \kappa \nabla^2 \bar{c}. \tag{3.2}$$

The fundamental problem of tracer turbulence is to determine the effect of the Reynolds fluxes $\overline{\boldsymbol{u}' c'}$ on the mean tracer concentration.

In order to determine the Reynolds fluxes, we must solve the equation for the tracer fluctuations,

$$c'_t + \bar{\boldsymbol{u}} \cdot \nabla c' + \nabla \cdot \left[\boldsymbol{u}' c' - \overline{\boldsymbol{u}' c'} \right] - \kappa \nabla^2 c' = -\boldsymbol{u}' \cdot \nabla \bar{c}. \tag{3.3}$$

Inspection of eq. (3.3) further suggests that if $c' = 0$ at $t = 0$ then, c' and $\nabla \bar{c}$ are linearly related at all times. The equation for the tracer fluctuations is linear in c' for a prescribed velocity field and it is forced by $\nabla \bar{c}$: doubling the forcing doubles the tracer fluctuations. In other words we are stating that tracer fluctuations c' are generated by advective distortion of the mean gradient, $\nabla \bar{c}$. It follows that the eddy flux $\overline{\boldsymbol{u}' c'}$ is linearly related to the mean gradient $\nabla \bar{c}$. The coefficient of proportionality between the tracer flux and the mean tracer gradient turns out to be the eddy diffusivity introduced in the previous lecture.

3.2 Eddy fluxes and the multi-scale method

Using an assumption of scale separation between the turbulent eddies and the mean, we will show how the concept of a tracer diffusivity arises in a Eluerian context. The scale separation approach is based on the assumption that the advected tracer is weakly inhomogeneous on a scale L much greater than the scale l_0 of the turbulent fluctuations. The goal is then to derive equations for the evolution of the coarse-grained averaged tracer on the length scale L, and on a time scale T large compared with the time scale $t_0 \sim l_0/u_0$, characteristic of the energy containing eddies of the turbulence. We may then choose an intermediate scale λ, *e.g.* $\lambda = (l_0 L)^{1/2}$, and an intermediate time τ, *e.g.* $\tau = (t_0 T)^{1/2}$, satisfying,

$$l_0 \ll \lambda \ll L, \qquad t_0 \ll \tau \ll T, \tag{3.4}$$

and think of the overbar average as a "local average" over a cube of side λ, and a time of order τ. Averaged quantities will vary only on larger scales of order L and longer times of order T. The replacement of ensemble and volume/time averages is possible only if the turbulence is homogeneous and stationary on the small spatial and temporal scales, so that an ergodic assumptions can be made.

The multi-scale method that we use here was first introduced by Papanicolaou and Pirroneau (1981). The scale separation assumption suggests that a perturbation expansion can be done in terms of the small parameter $\epsilon \equiv l_0/L$. Suppose now that $c(\boldsymbol{x}, 0)$ is slowly varying so that,

$$c(\boldsymbol{x}, 0) = \mathcal{C}_0(\epsilon \boldsymbol{x}). \tag{3.5}$$

Eq. (3.1), together with the initial condition in eq. (3.5) suggests a multiple scale analysis with the slow variables,

$$\boldsymbol{X} = \epsilon \boldsymbol{x}, \boldsymbol{X}_2 = \epsilon^2 \boldsymbol{x}, \qquad T = \epsilon t, T_2 = \epsilon^2 t. \tag{3.6}$$

The solution of eq. (3.1) then takes the form,

$$c(\boldsymbol{x}, t; \epsilon) = C_0(\boldsymbol{X}, T) + \epsilon\, C_1(\boldsymbol{x}, t; \boldsymbol{X}, T) + \epsilon^2 C_2(\boldsymbol{x}, t; \boldsymbol{X}, T) + \ldots. \tag{3.7}$$

The quantity of interest is the large-scale, long-time, averaged field $\bar{c} = C_0(\boldsymbol{X}, T) + O(\epsilon)$. Its evolution is obtained by usual asymptotic methods. Substituting the expansion (3.7) into the advection diffusion equation (3.1), one obtains a series of equations order by order in ϵ.

The advecting velocity field must be expanded as well. We will assume that the velocity field is composed of a mean flow varying on the slow variables only $\boldsymbol{U}(\boldsymbol{X}, T)$ and a turbulent perturbations $\boldsymbol{u}'$,

$$\boldsymbol{u} \equiv \boldsymbol{U}(\boldsymbol{X}, T) + \boldsymbol{u}'(\boldsymbol{x}, t; \boldsymbol{X}, T). \tag{3.8}$$

We do not assume that the mean flow is of small amplitude compared to the turbulent flow as it is often done in the literature of eddy mean flow interactions

3.2.1 Zeroth order solution

Let's write the series of equations order by order in ϵ. At fist order we have,

$$\boldsymbol{O(\epsilon^0)}: \qquad C_{0t} + (\boldsymbol{U} + \boldsymbol{u}') \cdot \nabla_{\boldsymbol{x}} C_0 - \kappa \nabla^2_{\boldsymbol{x}} C_0 = 0. \tag{3.9}$$

The solution to this equation, satisfying the assumption that the initial tracer concentration is smooth, has the general form,

$$C_0 = \mathcal{C}_0(\boldsymbol{X}, T, \boldsymbol{X}_2, T_2). \tag{3.10}$$

3.2.2 First order solution

$$\boldsymbol{O(\epsilon)}: \qquad C_{1t} + (\boldsymbol{u}' + \boldsymbol{U}) \cdot \nabla_{\boldsymbol{x}} C_1 - \kappa \nabla^2_{\boldsymbol{x}} C_1 = -C_{0T} - (\boldsymbol{u}' + \boldsymbol{U}) \cdot \nabla_{\boldsymbol{X}} C_0. \tag{3.11}$$

Averaging over the small and fast scales scales, we have,

$$C_{0T} + \boldsymbol{U} \cdot \nabla_{\boldsymbol{X}} C_0 = 0, \tag{3.12}$$

from which it follows that,

$$C_{1t} + (\boldsymbol{u}' + \boldsymbol{U}) \cdot \nabla_{\boldsymbol{x}} C_1 - \kappa \nabla^2_{\boldsymbol{x}} C_1 = -\boldsymbol{u}' \cdot \nabla_{\boldsymbol{X}} C_0. \tag{3.13}$$

Solutions to this problem can be written in the form $C_1 = -(\boldsymbol{\xi} \cdot \nabla) C_0 + \mathcal{C}_1(\boldsymbol{X}, T)$, with $\boldsymbol{\xi}(\boldsymbol{x}, t)$ satisfying the equation,

$$\boldsymbol{\xi}_t + (\boldsymbol{U} + \boldsymbol{u}') \cdot \nabla_{\boldsymbol{x}} \boldsymbol{\xi} - \kappa \nabla^2_{\boldsymbol{x}} \boldsymbol{\xi} = \boldsymbol{u}'. \tag{3.14}$$

This equation resembles the equation for a particle displacement, except for the presence of the molecular diffusive term. This difference is extremely important, because molecular diffusion is ultimately the only process that can mix the tracer. Any theory of diffusion which neglects molecular processes must be taken with suspicion. The terms $\mathcal{C}_1(\boldsymbol{X}, T)$ represents a small correction to the initial tracer concentration.

3.2.3 Second order solution

$$\begin{aligned}
\boldsymbol{O(\epsilon^2)}: \qquad & C_{2t} + (\boldsymbol{u}' + \boldsymbol{U}) \cdot \nabla_{\boldsymbol{x}} C_2 - \kappa \nabla^2_{\boldsymbol{x}} C_2 = && (3.15)\\
& -C_{1T} - (\boldsymbol{u}' + \boldsymbol{U}) \cdot \nabla_{\boldsymbol{X}} C_1 + 2\kappa \nabla_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{X}} C_1 + && (3.16)\\
& \kappa \nabla^2_{\boldsymbol{X}} C_0 - C_{0T_2} - (\boldsymbol{U} + \boldsymbol{u}') \cdot \nabla_{\boldsymbol{X}_2} C_1. && (3.17)
\end{aligned}$$

By taking the large scale and long time average of this equation, we obtain the solvability condition,

$$C_{0T_2} + \boldsymbol{U} \cdot \nabla_{\boldsymbol{X}_2} C_0 + C_{1T} + \boldsymbol{U} \cdot \nabla_{\boldsymbol{X}} C_1 \;=\; \kappa \nabla^2_{\boldsymbol{X}} C_0 + \overline{(\boldsymbol{u}' \cdot \nabla_{\boldsymbol{X}})\, \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{X}} C_0}. \tag{3.18}$$

The rhs represents the diffusion of tracer concentration by molecular and turbulent processes. This is best seen if we write,

$$\begin{aligned}
\kappa \nabla^2_{\boldsymbol{X}} C_0 + \overline{(\boldsymbol{u}' \cdot \nabla_{\boldsymbol{X}})\, \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{X}} C_0} &= \nabla_{\boldsymbol{X}} \cdot \left[\kappa \nabla_{\boldsymbol{X}} C_0 + \overline{\boldsymbol{u}' \boldsymbol{\xi}} \cdot \nabla_{\boldsymbol{X}} C_0 \right] && (3.19)\\
&= \nabla_{\boldsymbol{X}} \cdot [\boldsymbol{D} \nabla_{\boldsymbol{X}} C_0], && (3.20)
\end{aligned}$$

where the tensor $\boldsymbol{D}$,

$$D_{ij} = \kappa \delta_{ij} + \overline{u'_i \xi_j}, \tag{3.21}$$

is the effective diffusivity tensor.

Summing the solvability conditions at $O(\epsilon)$ and $O(\epsilon^2)$ we obtain the evolution equation for the mean tracer concentration,

$$\bar{c}_t + \bar{\boldsymbol{u}} \cdot \nabla \bar{c} = \partial_{x_i}[D_{ij}\partial_{x_j}\bar{c}]. \tag{3.22}$$

The mean tracer concentration is given by $\bar{c} = \mathcal{C}_0 + \epsilon\mathcal{C}_1$ and derivatives include variations on the slow space and time of first seond order. The effective diffusivity $\boldsymbol{D}$ is obtained by solving eq. (3.14) and computing the correlations between $\boldsymbol{\xi}$ and the velocity fluctuations $\boldsymbol{u}'$.

3.2.4 Eddy diffusivity

The main result of the multiple scale analysis is that there is a relationship between the eddy flux $\overline{\boldsymbol{u}'c'}$ and the mean tracer gradient $\nabla\bar{c}$,

$$\overline{u_i'c'} = -D_{ij}\partial_{x_j}\bar{c}. \tag{3.23}$$

We want to understand what is the meaning of this diffusivity. Let us decompose diffusivity tensor into its symmetric and antisymmetric components,

$$D = D^s + D^a, \tag{3.24}$$

where the symmetric component is,

$$D^s_{ij} \equiv \frac{1}{2}\left(D_{ij} + D_{ji}\right), \tag{3.25}$$

and the antisymmetric component is,

$$D^a_{ij} \equiv \frac{1}{2}\left(D_{ij} - D_{ji}\right). \tag{3.26}$$

Using the equation for $\boldsymbol{\xi}$,

$$\overline{u_i'\xi_j} \quad = \quad \overline{\left(\xi_{i,t} + u_k'\xi_{i,k} + U_k\xi_{i,k} - \kappa\xi_{i,kk}\right)\xi_j}, \tag{3.27}$$

we obtain useful expressions for the diffusive and skew components of the diffusivity,

$$\begin{aligned} D^s_{ij} &= \frac{1}{2}\partial_t\overline{(\xi_i\xi_j)} + \frac{1}{2}\partial_{x_k}\left[\overline{(u_k' + U_k)\left(\xi_i\xi_j\right)} - \kappa\partial_{x_k}\overline{(\xi_i\xi_j)}\right] + \kappa\overline{\xi_{i,k}\xi_{j,k}} \\ D^a_{ij} &= \frac{1}{2}\overline{\xi_j\xi_{i,t} - \xi_i\xi_{j,t}} + \frac{1}{2}\left[\overline{\xi_j\left(u_k' + U_k\right)\xi_{i,k} - \xi_i\left(u_k' + U_k\right)\xi_{j,k}} - \kappa\partial_{x_k}\overline{(\xi_{i,k}\xi_j - \xi_{j,k}\xi_i)}\right] \end{aligned} \tag{3.28}$$

The first line is symmetric and represents the diffusive component. The second line is asymmetric and is the skew component. Molecular diffusion contributes a positive definite term to the diffusive component.

3.2.5 Antisymmetric component of the eddy diffusivity

The role of the antisymmetric component of the diffusivity tensor is best interpreted if we write $\boldsymbol{D}^a$ in the form,

$$D^a_{ij} = -\epsilon_{ijk}\Psi_k. \tag{3.29}$$

This is the generic form of an antisymmetric tensor in three dimensions, i.e. any antisymmetric tensor can be written in the form (3.29). The tracer flux associated to the antisymmetric component of the diffusivity is the so called *skew flux*,

$$\overline{\boldsymbol{u}'c'}^a = -D^a_{ij}\partial_{x_j}\bar{c} = \epsilon_{ijk}\Psi_k\partial_{x_j}\bar{c} = \boldsymbol{\Psi}\times\nabla\bar{c}. \tag{3.30}$$

The skew flux can also be written as,

$$\nabla\cdot\overline{\boldsymbol{u}'c'}^a = \nabla\times(\boldsymbol{\Psi}\bar{c}) - (\nabla\times\boldsymbol{\Psi})\cdot\nabla\bar{c} = \nabla\times\boldsymbol{\Psi}\bar{c} + \bar{\boldsymbol{u}}_S\cdot\nabla\bar{c}, \tag{3.31}$$

where,

$$\bar{\boldsymbol{u}}_S = -\nabla\times\boldsymbol{\Psi}. \tag{3.32}$$

The skew flux is therefore equal to the sum of a rotational flux and an advective flux. The rotational flux does not affect the tracer evolution, since it has no divergence. The advective component represent the transport of tracer by the Stokes drift $\bar{\boldsymbol{u}}_S$.

3.2.6 Symmetric component of the eddy diffusivity

Consider the evolution of the variance of the mean tracer concentration,

$$\partial_t\bar{c}^2 + \nabla\cdot\left(\bar{\boldsymbol{u}}\bar{c}^2 + \bar{\boldsymbol{u}}_S\bar{c}^2 + \boldsymbol{D}^s\nabla\bar{c}^2\right) \;=\; -D^s_{ij}\partial_{x_i}\bar{c}\;\partial_{x_j}\bar{c}. \tag{3.33}$$

Changes in tracer variance can be induced by the terms inside the divergence or the term in the rhs. The terms inside the divergence depends both on the antisymmetric and the symmetric components of the eddy diffusivity. The term on the rhs instead depends on the symmetric component only. The terms inside the divergence represent processes that move tracer variance around the fluid without changing the total integral of the tracer variance. The rhs term, instead, represent a net source of tracer variance. hence only the symmetric component of the eddy diffusivity represent processes that enhance/reduce tracer fluctuations.

Let's compute the term on the rhs of equation (3.33),

$$-D^s_{ij}\partial_{x_i}\bar{c}\;\partial_{x_j}\bar{c} = -\frac{1}{2}\partial_t\overline{(\xi_i\partial_{x_i}\bar{c})^2} - \partial_{x_k}\left[\overline{(u_k+U_k)\,(\xi_i\partial_{x_i}\bar{c})^2} - \kappa\partial_{x_k}\overline{(\xi_i\partial_{x_i}\bar{c})^2}\right] - \kappa\overline{(\xi_{i,k}\partial_{x_i}\bar{c})^2}.$$

Integration over the fluid volume, we obtain,

$$-\int\int\int \mathrm{d}x\mathrm{d}y\mathrm{d}z\ D^s_{ij}\partial_{x_i}\bar{c}\ \partial_{x_j}\bar{c} = -\int\int\int \mathrm{d}x\mathrm{d}y\mathrm{d}z\ \left[\frac{1}{2}\partial_t\overline{(\xi_i\partial_{x_i}\bar{c})^2} + \kappa\overline{(\xi_{i,k}\partial_{x_i}\bar{c})^2}\right] \leq 0.$$

The molecular processes and transience terms that appear in the symmetric component of the eddy diffusivity tend to reduce the tracer variance, i.e. they tend to homogenize the mean tracer concentration.

3.2.7 Eddy diffusivity and moments of tracer concentration

An alternative approach to understand the effect of the eddy diffusivity on the mean tracer concentration is to compute the evolution of the tracer moments. The first three tracer moments are defined as,

$$M = \int\int\int \mathrm{d}x\mathrm{d}y\mathrm{d}z\ \bar{c}, \tag{3.34}$$

$$M_i = \int\int\int \mathrm{d}x\mathrm{d}y\mathrm{d}z\ x_i\bar{c}, \tag{3.35}$$

$$M_{ij} = \int\int\int \mathrm{d}x\mathrm{d}y\mathrm{d}z\ x_ix_j\bar{c}. \tag{3.36}$$

The evolution of the tracer moments is given by,

$$\frac{dM}{dt} = 0, \tag{3.37}$$

$$\frac{dM_i}{dt} = \int\int\int \mathrm{d}x\mathrm{d}y\mathrm{d}z\ \left(\bar{u}_i + \bar{u}_{Si} + \partial_j D^s_{ij}\right)\bar{c}, \tag{3.38}$$

$$\frac{dM_{ii}}{dt} = 2\int\int\int \mathrm{d}x\mathrm{d}y\mathrm{d}z\ \left(\bar{u}_i + \bar{u}_{Si} + \partial_j D^s_{ij}\right)x_i\bar{c} + 2\int\int\int \mathrm{d}x\mathrm{d}y\mathrm{d}z\ D^s_{ii}\bar{c}. \tag{3.39}$$

Consider a tracer patch with concentration different from zero only in a small area around $x_i = 0$. Then we can approximate the moment evolution as,

$$\frac{dM}{dt} = 0, \tag{3.40}$$

$$\frac{dM_i}{dt} \approx \left(\bar{u}_i + \bar{u}_{Si} + \partial_j D^s_{ij}\right)M, \tag{3.41}$$

$$\frac{dM_{ii}}{dt} \approx 2D^s_{ii}M. \tag{3.42}$$

3.2.8 Summary about eddy diffusivity

In summary then,

- The symmetric part of the diffusivity tensor corresponds to something like diffusive transport.
- The antisymmetric part, which is almost never zero, and it is in fact usually dominant for rotationally dominated waves, corresponds to an advective transport. As a result, the mean advecting velocity that appears in eq. (3.22) is not $\bar{\boldsymbol{u}}$, but the velocity $\bar{\boldsymbol{u}} + \bar{\boldsymbol{u}}_S$. This seems to be telling us that the Eulerian mean velocity $\bar{\boldsymbol{u}}$ is not the most natural choice of "mean".

3.3 The Transformed Eulerian Mean

The TEM formalism hinges on a transformation of the mean tracer equation to decompose the eddy flux in an advective and a diffusive components analogous to the Stokes drift and diffusivity encountered above. The TEM decomposition, however, does not require a scale separation assumption. Consider a tracer of concentration c, that satisfies a conservation equation of the form,

$$c_t + \boldsymbol{u} \cdot \nabla c = S\{c\}, \tag{3.43}$$

where $S\{c\}$ represents sources and sinks of tracer. The velocity field is assumed to be divergenceless, i.e. $\nabla \cdot \boldsymbol{u} = 0$. Following a Reynolds decomposition of variables into a slowly changing mean and fluctuations, the mean conservation budget for the tracer c is,

$$\bar{c}_t + \bar{\boldsymbol{u}} \cdot \nabla \bar{c} = -\nabla \cdot \boldsymbol{F}\{c\} + \bar{S}\{c\}, \tag{3.44}$$

where $\boldsymbol{F}\{c\} = \overline{\boldsymbol{u}'c'}$ is the eddy flux of c. The manipulation of such equations to obtain a transformed set hinges on the introduction of a nondivergent *residual circulation* $\bar{\boldsymbol{u}}^\dagger \equiv \bar{\boldsymbol{u}} + \nabla \times \boldsymbol{\Psi}_T$, where the choice of the vector streamfunction $\boldsymbol{\Psi}_T$ is left for the moment open. In terms of the residual circulation the budget in (3.44) becomes,

$$\bar{c}_t + \bar{\boldsymbol{u}}^\dagger \cdot \nabla \bar{c} = -\nabla \cdot \boldsymbol{F}^\dagger\{c\} + \bar{S}\{c\}, \tag{3.45}$$

where we introduced the *residual flux*,

$$\boldsymbol{F}^\dagger\{c\} \equiv \boldsymbol{F}\{c\} - \boldsymbol{\Psi} \times \nabla \bar{c}. \tag{3.46}$$

We are free to drop the term $\nabla \times (\boldsymbol{\Psi}_T \bar{c})$ in the definition of $\boldsymbol{F}^\dagger\{c\}$, because this term is non-divergent and does not affect the evolution of $\bar{c}$.

The residual flux and the full flux have the same projection along the mean tracer gradient,

$$\boldsymbol{F}^\dagger\{c\} \cdot \nabla \bar{c} = \boldsymbol{F}\{c\} \cdot \nabla \bar{c}, \tag{3.47}$$

but have different projections in the plane orthogonal to $\nabla \bar{c}$. Thus the vector streamfunction $\boldsymbol{\Psi}$ can be chosen in such a way as to eliminate the flux component in the plane orthogonal to $\nabla\bar{c}$, the so-called *skew flux*, leaving just the flux component along $\nabla\bar{c}$ to remain in $\boldsymbol{F}^{\dagger}\{c\}$. This manipulation has advantages from a modeling perspective, because the parameterization problem is simpler for the down-gradient component of a flux, than for the skew component.

The decomposition of the eddy flux into components across and along the mean tracer gradient is given by,

$$\boldsymbol{F}\{c\} = \frac{\overline{\boldsymbol{u}'c'}\cdot\nabla\bar{c}}{|\nabla\bar{c}|^2}\nabla\bar{c} - \frac{\overline{\boldsymbol{u}'c'}\times\nabla\bar{c}}{|\nabla\bar{c}|^2}\times\nabla\bar{c}. \tag{3.48}$$

The cross-gradient component is equivalent to a mean tracer advection, since,

$$\nabla\cdot\left[\frac{\overline{\boldsymbol{u}'c'}\times\nabla\bar{c}}{|\nabla\bar{c}|^2}\times\nabla\bar{c}\right] = \left[\nabla\times\frac{\overline{\boldsymbol{u}'c'}\times\nabla\bar{c}}{|\nabla\bar{c}|^2}\right]\cdot\nabla\bar{c}. \tag{3.49}$$

This term is the skew flux. The choice of $\boldsymbol{\Psi}$ that removes the full skew component from the residual flux is then,

$$\boldsymbol{\Psi}_T = -\frac{\overline{\boldsymbol{u}'c'}\times\nabla\bar{c}}{|\nabla\bar{c}|^2}. \tag{3.50}$$

With this definition the residual circulation and residual buoyancy flux are,

$$\bar{\mathbf{u}}^{\dagger} = \bar{\boldsymbol{u}} + \nabla\times\boldsymbol{\Psi}, \qquad \boldsymbol{F}^{\dagger}\{c\} = \frac{\overline{\boldsymbol{u}'c'}\cdot\nabla\bar{c}}{|\nabla\bar{c}|^2}\nabla\bar{c}. \tag{3.51}$$

The mean tracer variance budget, together with the eddy tracer variance budget, suggests what the form of the residual flux must be,

$$\partial_t\bar{c}^2 + \nabla\cdot\left[\bar{\boldsymbol{u}}\bar{c}^2 + \boldsymbol{\Psi}_T\times\nabla\bar{c}^2 - 2\boldsymbol{F}^{\dagger}\{c\}\ \bar{c}\right] = 2\boldsymbol{F}^{\dagger}\{c\}\cdot\nabla\bar{c}, \tag{3.52}$$

$$\partial_t\overline{c'^2} + \nabla\cdot\left[\overline{\boldsymbol{u}c'^2} - \kappa\nabla\overline{c'^2}\right] = -2\boldsymbol{F}^{\dagger}\{c\}\cdot\nabla\bar{c} - \kappa\overline{|\nabla c'|^2} \tag{3.53}$$

$$\tag{3.54}$$

where we neglected diffusive fluxes in the mean budgets. If the eddy variance equation is steady and homogeneous, then $\boldsymbol{F}^{\dagger}\{c\}\cdot\nabla\bar{c}\le 0$ suggesting a downgradient diffusive closure for the residual flux.

This definition of residual streamfunction is generally different from the Stokes streamfunction. We can see this plugging the expression for the eddy flux derived in the previous section,

$$\overline{u_i'c'} = \overline{u_i'\xi_j'}\partial_{x_j}\bar{c} = \epsilon_{ijk}\Psi_k\partial_{x_j}\bar{c} + D^s_{ij}\partial_{x_j}\bar{c}, \tag{3.55}$$

into the definition of $\boldsymbol{\Psi}_T$,

$$\boldsymbol{\Psi}_T = \boldsymbol{\Psi} - \frac{(\boldsymbol{\Psi}\cdot\nabla\bar{c})\nabla\bar{c}}{|\nabla\bar{c}|^2} - \frac{(D^s\nabla\bar{c})\times\nabla\bar{c}}{|\nabla\bar{c}|^2}. \tag{3.56}$$

The residual streamfunction differs from the Stokes drift streamfunction in two respects. First we ignore any Stokes drift along tracer surfaces, because it does not advect tracer. Second we add to the streamfunction any diffusive term that is directed along tracer contours, because it does not affect the tracer variance budget. This additional term represents a drift of the baricenter of a tracer patch due to variations in the rate of diffusion along a tracer isosurface.

In geophysical flow it is often the case that eddy fluxes are directed along isentropic surface, i.e. the eddy fluxes are adiabatic. In this case it is accurate to assume that the diffusive flux is down the mean gradient of the tracer $\bar{c}$ projected along the entropy surface,

$$D^s \nabla \bar{c} = -K \left[\nabla \bar{c} - \frac{\nabla \bar{c} \cdot \nabla \bar{b}}{|\nabla \bar{b}|^2} \nabla \bar{b} \right]. \tag{3.57}$$

Substituting this expression in (3.56), we obtain,

$$\frac{(D^s \nabla \bar{c}) \times \nabla \bar{c}}{|\nabla \bar{c}|^2} = K \frac{(\nabla \bar{c} \cdot \nabla \bar{b})(\nabla \bar{c} \times \nabla \bar{b})}{|\nabla \bar{b}|^2 |\nabla \bar{c}|^2}. \tag{3.58}$$

This term clearly vanishes if one uses the tracer entropy $\bar{b}$ to define the residual streamfunction. For other tracers, it is proportional to the angle θ between the entropy and tracer isosurfaces,

$$\left| \frac{(D^s \nabla \bar{c}) \times \nabla \bar{c}}{|\nabla \bar{c}|^2} \right| = \frac{1}{2} K \sin 2\theta. \tag{3.59}$$

3.4 Diffusive flux in tracer coordinates

The Transformed Eulerian Mean decomposition shows that the turbulent eddies affect the evolution of the mean tracer concentration through advective and diffusive components. The advective component depends on the skew eddy flux, i.e. the flux along tracer contours. This component is hard to predict because it depends on the details of the correlations between eddy velocities and displacements. The diffusive component, instead, has a simpler behavior. It tends to spread tracer concentration. While the spatial variations of the eddy diffusivity are key to predict the advective component, the magnitude of the diffusive component is all that is needed to predict the correct destruction of tracer variance. For all these reasons, it seems possible to derive closures for the diffusive component of the eddy flux, while closures for the advective component might be beyond our grasp. In this section, we will discuss techniques to estimate the diffusive component of the tracer flux. In the next lecture, we will show how to make the advective component of the tracer flux part of the prognostic variables, so that we do not need to parameterize it.

Let us consider our 3D tracer field. We assume that instantaneous iso-surfaces of tracer are closed (though the theory works also if they encounter boundaries). We wish to express the tracer equation in tracer coordinates. In this framework, advection will become part of the coordinate system: advection moves tracer surfaces around. The only remaining flux is the eddy transport across tracer contours, i.e. the diffusive tracer flux. The idea was first introduced by Noboru Nakamura (Journal of Atmospheric Sciences, 52, 2096-2108).

The key coordinate labeling a surface of constant c is its enclosed volume,

$$V(C,t) = \iiint_{c \leq C} \mathrm{d}V = \int_{C_{min}}^{C} \mathrm{d}C' \oiint_{c=C'} |\nabla c|^{-1} \mathrm{d}A, \tag{3.60}$$

where $\mathrm{d}A$ is the area element along a c-surface. The last step of equation (3.62) essentially divides the volume in a series of shells defined by the closed C-surfaces. The volume of each cell is given by the integral over the full surface A of its thickness $\mathrm{d}c/|\nabla c|$. For definiteness, we assume that the closed region is a minimum of c, so that A is a monotonically increasing function of C. Note that this integral can include multiply connected volumes, if there are multiple blobs enclosing the same tracer concentration.

The volume density in c-space, i.e. the volume between two tracer iso-surfaces is,

$$\frac{\partial V}{\partial C} = \oiint_{c=C'} |\nabla c|^{-1} \mathrm{d}A. \tag{3.61}$$

Analogously, we the volume integral of any quantity π is defined as,

$$\Pi(C,t) = \iiint_{c \leq C} \chi \mathrm{d}V = \int_{C_{min}}^{C} \mathrm{d}C' \oiint_{c=C'} |\nabla c|^{-1} \pi \mathrm{d}A, \tag{3.62}$$

and its volume integral between two iso-surfaces is,

$$\frac{\partial \Pi}{\partial C} = \oiint_{c=C} |\nabla c|^{-1} \pi \mathrm{d}A. \tag{3.63}$$

Consider now the time evolution of the volume enclosed within a tracer surface,

$$\frac{\partial V}{\partial t}(C,t) = \oiint_{c=C} \frac{\boldsymbol{u}_c \cdot \nabla c}{|\nabla c|} \mathrm{d}A, \tag{3.64}$$

where $\boldsymbol{u}_c$ is the speed of the tracer surface $c = C$, i.e. $\partial_t c + \boldsymbol{u}_c \cdot \nabla c = 0$. Hence we can also write,

$$\frac{\partial V}{\partial t}(C,t) = -\oiint_{c=C} |\nabla c|^{-1} \frac{\partial c}{\partial t} \mathrm{d}A = -\frac{\partial}{\partial C} \iiint \frac{\partial c}{\partial t} \mathrm{d}A, \tag{3.65}$$

where we used the relationship in (3.63). Plugging the tracer budget,

$$c_t + \boldsymbol{u} \cdot \nabla c = \kappa \nabla^2 c. \tag{3.66}$$

into equation (3.65) gives,

$$\frac{\partial V}{\partial t}(C,t) = \frac{\partial}{\partial C}\iiint \nabla\cdot[\boldsymbol{u}c - \kappa\nabla c]\,\mathrm{d}V. \tag{3.67}$$

Using Stokes theorem,

$$\begin{aligned}
\frac{\partial V}{\partial t}(C,t) &= \frac{\partial}{\partial C}\oint\!\!\oint_{c=C}\frac{[\boldsymbol{u}c-\kappa\nabla c]\cdot\nabla c}{|\nabla c|}\mathrm{d}A, && (3.68)\\
&= \frac{\partial}{\partial C}\left(C\oint\!\!\oint_{c=C}\frac{\boldsymbol{u}\cdot\nabla c}{|\nabla c|}\mathrm{d}A\right) - \frac{\partial}{\partial C}\oint\!\!\oint_{c=C}\frac{\kappa\nabla c\cdot\nabla c}{|\nabla c|}\mathrm{d}A, && (3.69)\\
&= \frac{\partial}{\partial C}\left(C\iiint\nabla\cdot\boldsymbol{u}\mathrm{d}\;V\right) - \frac{\partial}{\partial C}\oint\!\!\oint_{c=C}\frac{\kappa\nabla c\cdot\nabla c}{|\nabla c|}\mathrm{d}A, && (3.70)\\
&= -\frac{\partial}{\partial C}\oint\!\!\oint_{c=C}\frac{\kappa\nabla c\cdot\nabla c}{|\nabla c|}\mathrm{d}A, && (3.71)\\
& && (3.72)
\end{aligned}$$

An expression for the mean tracer evolution can be now had simply by replacing the system $V = V(C,t)$ by one in which $C = C(V,t)$,

$$\frac{\partial C}{\partial t} = -\frac{\partial C}{\partial V}\frac{\partial V}{\partial t} = \frac{\partial}{\partial V}\oint\!\!\oint_{c=C}\frac{\kappa\nabla c\cdot\nabla c}{|\nabla c|}\mathrm{d}A. \tag{3.73}$$

Introducing the effective diffusivity K_{eff},

$$K_{eff} = \left(\frac{\partial C}{\partial V}\right)^{-1}\oint\!\!\oint_{c=C}\frac{\kappa\nabla c\cdot\nabla c}{|\nabla c|}\mathrm{d}A. \tag{3.74}$$

we have,

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial V}\left(K_{eff}\frac{\partial C}{\partial V}\right), \tag{3.75}$$

a diffusive equation for the mean tracer concentration. The effective diffusivity is positive definite and represents diffusion of tracer across iso-surfaces.

Notice that the effective diffusivity depends on the dissipation of tracer variance within two tracer surfaces,

$$K_{eff} = \left(\frac{\partial C}{\partial V}\right)^{-1}\frac{\partial}{\partial C}\iiint\kappa\nabla c\cdot\nabla c\;\mathrm{d}V \tag{3.76}$$

and it is therefore proportional to the diffusive flux of tracer over that volume.