Chapter 8

Statistical closures for isotropic homogeneous 3D turbulence

8.0.1 Parameterizations in the absence of a mean flow

Statistical closures constitute an intriguing alternative to conventional numerical simulations of the primitive dynamical equations of turbulence. The NavierStokes equation at high Reynolds number, for example, defies direct numerical computation, primarily because the solutions of this strongly nonlinear equation vary rapidly in both space and time. In contrast, statistical closures provide approximate descriptions of the average behavior of an ensemble of turbulent realizations; these statistical solutions are relatively smooth.

The construction of a statistical description of turbulence is far from unambiguous. The averaging of a nonlinear equation leads to an infinite hierarchy of moment equations that is usually closed by adopting some approximate relation between high-order moments and low-order moments. We illustrate as an example the

We begin with the fundamental equation,

$$\left(\frac{\partial}{\partial t} + \nu_i\right)\hat{u}_i(t) = \int A_{ijk} \tag{8.1}$$

where $\hat{u}_i = \hat{u}(k_i, t)$ is one component of the Fourier transform of the velocity field. For simplicity we will consider only one component of the velocity field and we will forget about the fact that the Fourier components are complex.

The approach that we will take is to derive equations for the energy, i.e. the second order moment, and use a quasi-normal approximation to infer the relationship between higher order moments and energy. This closure scheme is referred to as an

Eddy Damped Quasi-Normal Closure. We refer the reader to chapter 5 of Salmon's book for a presentation of the EDQN closure.

8.0.2 Parameterizations in the presence of a mean flow

In geophysical flows, it is typically the case that there is some large scale flows driven by external forcing applied either at the boundaries or in the interior. Turbulence however appears at small scale as "noise" superimposed on the mean flow. Recall that we can separate the effects of mean and turbulence by taking a Reynolds average of the Navier-Stokes equations,

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho_0} \frac{\partial}{\partial x_j} \left(\overline{p} \delta_{i.j} - \rho_0 \nu \frac{\partial \overline{u}_i}{\partial x_j} + \rho_0 \overline{u'_j u'_i} \right), \qquad (8.2)$$

$$\frac{\partial \overline{u}_j}{\partial x_j} = 0, \qquad (8.3)$$

where as usual, the primes denote departure from the average. Notice that we are using an overbar to denote averages, because in this case the averages are not necessarily ensemble averages. In this section we associate the average flow with the large scale motion and the primed flow with the smaller scales. This constitutes our definition of average, if you like. Somewhat inconsistently we assumed that the averaging operator satisfies the all the usual properties for ensemble averages, like $\overline{\overline{u}_i u'_j} = 0$.

For the case of turbulence superimposed on a mean flow, the closure problem amounts to finding a way to express the the Reynolds stresses $\overline{u'_j u'_i}$ in terms of resolved quantities $(\overline{u}_i)$. This is a very different question from that considered above, when we considered the case of predicting the evolution of turbulence in the absence of a mean flow. Both questions are very interesting, but in most geophysical applications the problem in the presence of a mean flow is more relevant.

Before we proceed to derive a closure scheme, we want to build some intuition on the effect of turbulence on the mean flow. We follow an argument given by Salmon in chapter 3 of his book. Insight can be gained considering the energy budget for the mean flow. To form an equation for the energy in the large scale motions, we multiply (6.2) by $\overline{u}_i$ and integrate over the whole fluid. After integration by parts we get, equations,

$$\frac{d}{dt} \int \int \int \frac{1}{2} \overline{u}_i \overline{u}_i \ \mathrm{d}\boldsymbol{x} + \nu \int \int \int \frac{\partial \overline{u}_i}{\partial x_j} \frac{\partial \overline{u}_i}{\partial x_j} \ \mathrm{d}\boldsymbol{x} = -C, \qquad (8.4)$$

where,

$$C = - \int \int \int \overline{u'_i u'_j} \frac{\partial \overline{u}_i}{\partial x_j} \ \mathrm{d}\boldsymbol{x}, \qquad (8.5)$$

is the rate at which large-scale energy is being converted to small-scale energy by the advection terms in the momentum equations. To show this, you can form an equation for the rate of change of energy in small spatial scales; you would find that the term (6.5) appears with the opposite sign. Now we want to show that C is *typically* positive in three dimensional turbulence. The word typically is a remainder that all such statements are statements about statistical averages, and rest on assumptions about average behavior. Three dimensional turbulence is predictable only in a statistical sense and closures can be derived only for the average behavior, not for individual realizations.

Now consider the situation where the large scale flow is given by a time independent shear flow of the form,

$$\bar{\boldsymbol{u}} = (u(y), 0, 0).$$

(8.6)

Then C is given by,

$$C = -\int\int\int \overline{u'v'}\frac{\partial \bar{u}}{\partial y}\ \mathrm{d}\boldsymbol{x}.$$

(8.7)

Vortex stretching is the primary mechanism for energy transfer between different scales in three dimensional turbulence. Let us assume the small-scale turbulence as an initially isotropic collection of vortex tubes. Tube A is stretched by the mean shear, and the magnitude of its vorticity therefore increases. On the other hand, the vortex tube C is squashed, and the magnitude of its vorticity decreases. Tube B remains unstretched. At a later time, cortex tube A becomes the dominant contribution to the Reynolds stress, and it contributes negatively to $\overline{u'v'}$. The Reynolds momentum flux is down-gradients and C is positive. Small-scale three dimensional turbulence tends to slow down the large-scale flow.

The simplest closure scheme that captures the property that C is positive definite employs an eddy viscosity assumption,

$$\overline{u'_j u'_i} = -\nu_T\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

(8.8)

This is a **first order closure**. Models vary in the complexity of the system used to specify ν_T, the eddy viscosity. ν_T can be specified directly in terms of the large-scale quantities of the flow and/or model grid, or it can be specified in terms of subgridscale quantities for which extra prognostic equations are required. Most eddy viscosity models draw on Prandtl's **mixing length model**.

Mixing length model

The difficulty with pursuing a vorticity based argument is that it does not provide any guidance on how to pick the effective diffusivity. Prandtly proposed therefore a different argument to give a scaling for ν_T.

Consider a parcel in the shear flow introduced above, $(\bar{u}(y), 0, 0)$. Assume that the parcel was initially at at some position y. If the parcel moves due to turbulent motion, up to a position $y + \delta y$, and it conserves momentum, then it has a momentum deficit compared to the parcels around it of,

$$u' = [\bar{u}(y + \delta y) - \bar{u}(y)] + \delta u \approx \delta y \frac{\partial \bar{u}}{\partial y} + \delta u, \tag{8.9}$$

$$v' = -\delta v, \tag{8.10}$$

where δu and δv are the random velocity fluctuations that every particle experience. Notice that we also had to assume,

$$\delta y \ll \frac{\partial \bar{u}/\partial y}{\partial^2 \bar{u}/\partial y^2}, \tag{8.11}$$

in order to neglect higher order terms in the Taylor series expansion. If we further assume that the statistics of turbulent fluctuations are homogeneous and isotropic,

$$\overline{u'v'} = -\overline{\delta y \delta v} \frac{\partial \bar{u}}{\partial y}. \tag{8.12}$$

Introducing the **mixing length** l - the distance at which δv and δy become uncorrelated - we can write,

$$\overline{\delta y \delta v} = -c \, l \, \sqrt{\overline{\delta v^2}}, \tag{8.13}$$

where c is a constant. We then have,

$$\overline{u'v'} = -\nu_T \frac{\partial \bar{u}}{\partial y} = -c \, l \, \sqrt{\overline{\delta v^2}} \frac{\partial \bar{u}}{\partial y}, \tag{8.14}$$

where ν_T is the eddy viscosity,

$$\nu_T = c \, l \, \sqrt{\overline{\delta v^2}}. \tag{8.15}$$

Under the isotropy assumption $\overline{\delta v^2} = \overline{\delta u^2} = \overline{\delta w^2}$, we can write this as

$$\nu_T = c_\mu \, l \, \sqrt{q} \tag{8.16}$$

where $q/2$ is the small-scale turbulent kinetic energy, TKE, and c_μ is again a constant. Eq.(6.16) could also be obtained on dimensional grounds, by assuming the turbulent motion is characterized by a single velocity scale $\sqrt{q}$, and a single lengthscale l.

The problem of estimating ν_T is now reduced to one of estimating the TKE and the mixing length l. An expression for ν_T can be obtained by assuming that the small-scale motions satisfy the Kolmogorov inertial range scaling, $E(k) \sim \epsilon^{2/3} k^{-5/3}$. Then we can estimate q as,

$$q \sim \int_{\pi/l}^{\infty} \epsilon^{2/3} k^{-5/3} \, \mathrm{d}k \sim \epsilon^{2/3} l^{2/3}. \tag{8.17}$$

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Thus we can write,

$$l \sim \frac{q^{3/2}}{\epsilon} \tag{8.18}$$

and use this expression in eq. (6.16),

$$\nu_T = c_\mu \frac{q^2}{\epsilon}. \tag{8.19}$$

Hence we have two equivalent representations of ν_T in terms of turbulent quantities: eq.(6.16) and eq.(6.19). One or the other of these representations for ν_T form the basis of many eddy viscosity models. We still need to know either q and l or q and ϵ to have a full closure.

Notice that if q and l change in space, it seems that we have to abandon the assumption that the turbulence is homogeneous. But homogeneity is at the core of eddy mixing length arguments. A way out of this apparent inconsistency is to assume that turbulence is homogeneous on scales smaller than the model grid size and thus we can apply mixing length theory. However variations in turbulent levels appear on the much larger scales explicitly resolved by the model and this is why we need equations for q and l or q and ϵ.

A second issue of concern is that eddy mixing length theory should not be used for non-conserved quantities. If we assume that the average is carried on distances so short that pressure effects do not change momentum much, then we can apply eddy mixing length theory to momentum. However this is often done for models whose resolution is too coarse for this to be true. The only rationale to use a large effective viscosity in these cases is numerical: without a large ν_T coarse resolution models tend to be unstable.

Smagorinsky model

Smagorinsky proposed a simple argument to determine the eddy viscosity without having to derive separate equations for q, l, or ϵ. The argument goes as follows. We start by assuming a local production/dissipation balance for the TKE,

$$-\overline{u_i' u_j'} \frac{\partial \bar{u}_i}{\partial x_j} = \epsilon. \tag{8.20}$$

Using the expression,

$$-\overline{u_j' u_i'} = \nu_T \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) = 2\nu_T S_{ij}, \tag{8.21}$$

we can rewrite eq. (6.20) as,

$$2\nu_T S_{ij}^2 = \epsilon. \tag{8.22}$$

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We can now use eddy mixing length theory to relate this expression to an eddy viscosity,

$$\epsilon = \frac{q^{3/2}}{l}, \qquad \nu_T = c_\mu \frac{q^2}{\epsilon} \quad \Rightarrow \quad \epsilon = \frac{\nu_T^3}{c_\mu^3 \, l^4}. \tag{8.23}$$

Plugging these expressions in eq. (6.22), we obtain,

$$\nu_T = l_m^2 \sqrt{2 S_{ij}^2} \tag{8.24}$$

where the new mixing length is given by,

$$l_m = c_\mu^{3/4} l. \tag{8.25}$$

Now we just need to decide on what is an appropriate estimate for l_m, the mixing length. In the Smagorinsky model, l_m is assumed to be proportional to the largest length scale of the unresolved motion, i.e. the grid scale Δx in a numerical model. Then from eq.(6.24) we have,

$$\nu_T = C_S \Delta x^2 \sqrt{2 S_{ij}^2}. \tag{8.26}$$

The value of the constant C_S is chosen by assuming that the cutoff wavenumber $k_C = \pi/\Delta x$ lies within a $k^{-5/3}$ inertial range of a Kolmogorov-type energy spectrum. Therefore the transfer through the cut (also called the subgrid drain or subgrid dissipation) is,

$$\epsilon = 2\nu_T S_{ij}^2 = C_s \Delta x^2 (2 S_{ij}^2)^{3/2}, \tag{8.27}$$

and hence,

$$S_{ij}^2 = \epsilon^{2/3} (C_s \Delta x^2)^{-2/3}/2, \tag{8.28}$$

which is related to an expression deriving directly from the Kolmogorov spectrum,

$$S_{ij}^2 \sim \int_0^{\pi/\Delta x} k^2 E(k) \mathrm{d}k \sim \frac{3}{4} C_K \epsilon^{2/3} \left(\frac{\pi}{\Delta x} \right)^{4/3}, \tag{8.29}$$

where C_K is the Kolmogorov constant. (The choice of $\pi/\Delta x$ as the cutoff wavenumber in this integral is necessarily arbitrary, since the value depends on the form of discretization used. Lilly (1967) suggests it as the 'largest wavenumber unambiguously representable on a finite difference mesh'. Since changing the coefficient simply alters the Smagorinsky constant in a corresponding manner, we shall not discuss the question further.) From the above we must have,

$$C_S \approx \frac{1}{\pi^2} \left(\frac{2}{3 C_K} \right)^{3/2}. \tag{8.30}$$

It is useful here, since the Smagorinsky model is widely used in oceanography, to remind ourselves of the assumptions we have used to get here. We have assumed,

1. A local balance between shear production and dissipation of subgridscale kinetic energy, i.e. ignoring buoyant production and transport of TKE.

2. A single characteristic lengthscale Δx.

3. Δx is within an inertial range appropriate to isotropic homogeneous turbulence.

These assumptions will be most clearly violated when the model resolution is so coarse that the inertial range is not even partially resolved, when there are strong inhomogeneities in the turbulence, so that transport of TKE is important, when the subgridscale turbulence cannot be characterized as isotropic homogeneous turbulence (i.e. stratification and rotation are important) and when buoyancy is important in generating or removing TKE. Because this model is not appropriate when buoyancy is important on the subgrid scales, it is typically only used in coarser resolution ocean models for the horizontal components of viscosity and diffusivity. If the model resolution is finer, so that buoyancy production of kinetic energy is resolved (i.e. an LES model), it might be justified to assume a balance between shear production and dissipation for the smaller subgrid scales, and hence use the Smagorinsky model for vertical diffusivities/viscosities too.