Chapter 1

Introduction

Richard Feynman (1964) in his *Lecture on Physics* observed that,

> Often, people, in some unjustified fear of physics, say you can't write an equation for life. Well, perhaps we can. As a matter of fact, we very possibly already have the equation to a sufficient approximation, when we write the equation for quantum mechanics:
>
> $$H\psi = -\frac{\hbar}{i}\frac{\partial \psi}{\partial t}. \qquad (1.1)$$
>
> However, we are unable to reconstruct the field of biology from this equation, and we depend on detailed observation of biological phenomena.

Uriel Frisch (1995) in his book *Turbulence* points out that an analogous situation prevails in the study of turbulent flows. The equation, generally referred to as the Navier-Stokes equation, has been know since Navier (1827) and Stokes (1845),

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho_0}\nabla p + \nu\nabla^2\boldsymbol{u}, \qquad (1.2)$$

$$\nabla \cdot \boldsymbol{u} = 0, \qquad (1.3)$$

The Navier-Stokes equations probably contain all of turbulence. Yet it would be foolish to try to guess from these equations all the variety of regimes of turbulent flows without looking at experimental facts. The phenomena are almost as varied as in the realm of life. The flows shown in Fig. 1 are example of solutions of the Navier-Stokes equations, with modifications to account for rotation and density variations. But nobody knows how to derive these solutions from the equations themselves.

A good way to make contact with the world of turbulence phenomena is through observations of natural flows. Examples are ubiquitous in the ocean, atmosphere,
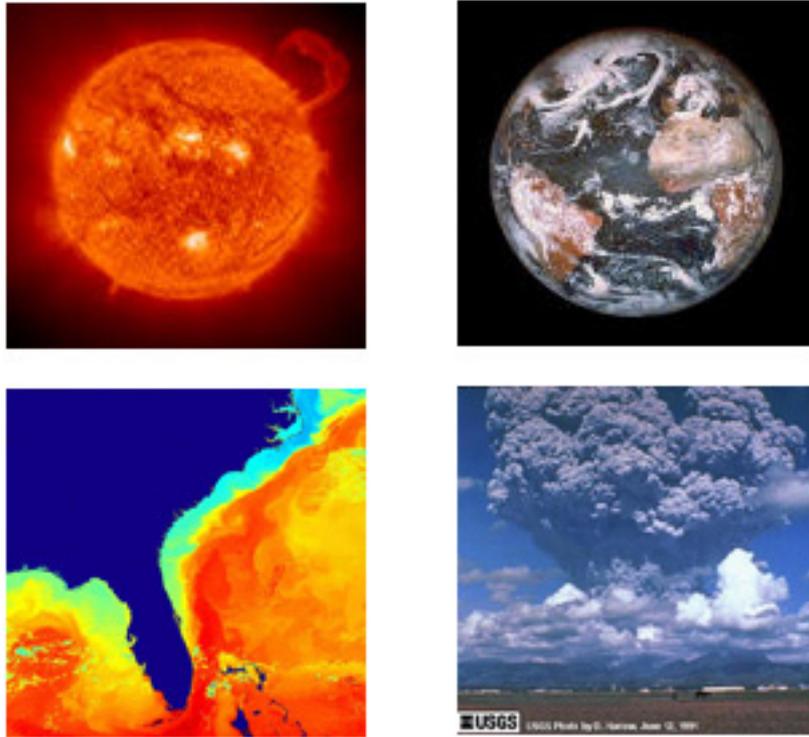
Figure 1.1: Examples of turbulent flows at the surface of the Sun, in the Earth's atmosphere, in the Gulf Stream at the ocean surface, and in a volcanic eruption.

lakes, and rivers of our Earth, in the atmospheres of other planes, in stars, galaxies, and space gases (neutral and ionized). A few examples are shown in Fig. 1. These flows are very irregular and do not display the regularity of the solutions of the Navier-Stokes equations that you studied in introductory courses in fluid dynamics. The field of turbulence can be defined as the attempt to bring together our understanding of the laws that govern fluid dynamics (the Navier-Stokes equations) with the irregular nature of real flows.

But how can we tell which natural flows are turbulent and which are not? As for the problem of defining life, there is no simple answer. A useful approach is to list what properties must be present to consider a flow turbulent.

1.1 Properties of turbulent flows

- **Broadband spectrum in space and time**

  Turbulent flows are characterized by structures on a broad range of spatial and temporal scales, even given smooth or periodic initial conditions and forcing. That is turbulent flows have a broadband spectrum both in frequency and wavenumber domains.

Figure removed due to copyright restrictions.
Citation: Van Dyke, Milton. *An album of fluid motion.* Stanford, CA:
Parabolic Press, 1982, p. 176. ISBN: 9780915760039.

Figure 1.2: High Re jet exiting from a nozzle.

Figure 1.3: Uniform flow incident on a cylinder at low Re. Uniform flow incident on two cylinders at high Re.

If L is the length scale of the largest motions and l is the length scale of the smallest motions in a flow, then a large range of spatial scales implies $L \gg l$. The scale l is typically the scale at which dissipation becomes important and removes energy from the flow. The scale L, instead, is set by the forcing mechanisms that set the large-scale flow. The ratio L/l is the Reynolds number Re, and $L \gg l$ implies that the Reynolds number be large. Turbulent flows have large Reynolds numbers (Fig. 1.2 and 1.3).

- **Dominated by advective nonlinearity**

  A field of non-interacting linear internal waves with many different frequencies and wavenumbers can also have a large range of length scales, but it is not turbulent. Why not? In a turbulent flow the different scales interact, through the nonlinear terms in the equations of motion. And these nonlinear interactions are responsible for the presence of structure on many different scales. Thus the broad band spectrum appears as a result of the internal dynamics. In a field of linear internal waves, instead, the broad band spectrum is generated by external controls like forcing, initial or boundary conditions (Fig. 1.4).

- **Unpredictable in space and time**

  Turbulent flows are predictable for only short times and short distances. Even though we know the equations that govern the evolution of the fluid, we cannot make predictions about the details of the flow due to its sensitive dependence on initial and boundary conditions. This sensitive dependence is once more a result of the strong nonlinearity of the flow. Predictability, however, can be recovered in a statistical sense, as we will see in lecture 2 (Fig. 1.5). The sensitive dependence on initial and boundary conditions is a fundamental property of *chaotic* systems. Are thus turbulence and chaos synonyms? No. Turbulent motions are indeed chaotic, but chaotic motions need not be turbulent. Chaos may involve only a small number of degrees of freedom, *i.e.* it can be narrow band in space and/or time. There are numerous examples of chaotic systems characterized by temporal complexity, but spatial simplicity, like the Lorenz's system. Another class of chaotic flows is represented by amplitude equations

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Figure 1.4: These internal wave photographs were taken by astronauts on board the space shuttle on Jan. 14, 1986. The picture shows the sea surface of the Eastern Pacific, around the Galapagos Islands, 600 miles off the coast of Ecuador. The sea surface coverage of a photograph is about of 75 km by 75 km. There is a clear difference between the wavy patterns of internal waves and the turbulent patterns of clouds.

that describe the slow time and large scale evolution of nearly monochromatic waves. Turbulence is different, because it is always complex both in space and time.

- **Time irreversible**

  Turbulent motions are not time reversible. As time goes on, turbulent motions tend to forget their initial conditions and reach some equilibrated state. Turbulence mixes stuff up, it does not unmix it (Fig. 1.6). A challenge of this course will be to explain how irreversibility can arise in fluids that are governed by classical mechanics, *i.e.* Newton's dynamics, which is time reversible.

The classification of properties that a flow must display to be considered turbulent is a subject of continuous debate in the scientific community. Many authors make narrower definitions of turbulence, limiting the scope to

- flows exhibiting explosive three dimensional vortex stretching

- flows obeying Kolmogorov's cascade law (to be described later)

- flows with a finite cascade of energy toward smaller scales.

These definitions are arbitrarily exclusive, since there are many geophysical flows which share the fundamental properties of broadband spectrum, advective nonlinearity, unpredictability, and time irreversibility, yet, due to the effects of rotation and stratification, are not fully three dimensional, do not satisfy Kolmogorov's law, and have no energy transfer toward smaller scales.

The properties of turbulent flows are better described in terms of dualities between regular and turbulent flows. Here we follow a list proposed by Jim McWilliams, a very influential oceanographer from UCLA.

- Deterministic vs random.
  The equation of motions are deterministic, but their solutions have many attributes of random processes.

- Orderly versus chaotic.
  Many of the spatial and temporal patterns are orderly, but the overall behavior is clearly chaotic.

- Time reversible versus time irreversible.
  If we neglect ν and κ (because Re is large), then the equations are time reversible, *i.e.* any solutions for $(t, \boldsymbol{u}, p, b)$ is also a solution for $(-t, -\boldsymbol{u}, p, b)$, but the outcome is evidently irreversible, even after a short period of time of the order of the eddy turnover time, L/U. The breaking of symmetries is a salient feature of turbulent motions.

Figure 1.5: Two sections of one second of a signal recorded by a hot-wire (sampled at 5 kHz) in a wind tunnel. The two sections differ in some small details of the flow upstream, i.e. initial conditions. The statistical properties of the two signals are similar, but the details of the flow are completely different.
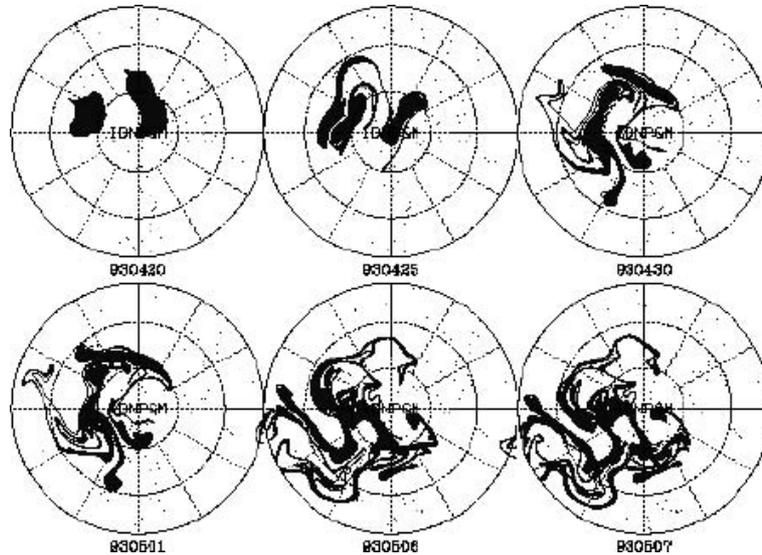
Figure 1.6: Evolution of a simulated tracer deployed in the Arctic vortex in April 1993. The circles on plots for 30 April, 1, 6, and 7 May are the location where tracer values representative of vortex air where measured aboard the ER-2 aircraft. There is good agreement between the location of the filaments of ex-vortex air in the simulation and the locations where vortex air was observed (See Waugh et al., 1997 for details). The tracer tends to spread out from the location where it was released.

- Materially confined versus dispersive.
  Parcels initially close to each other, tend to spread in time. Turbulence mixes tracers, it doesn't unmix them.

- Conservative versus non-conservative.
  If we neglect ν and κ, then the equations are conservative (of energy, tracer variance, etc.), but the outcome of turbulence is dissipative (the second law of thermodynamics applies). If dissipation continues to occur in a regime with only large-scale forcing or initial conditions – as is true for the Earth's climate system – then there must be a continuing transfer of fluid properties from larger spatial scales to smaller ones, and eventually small enough so that diffusive terms can be comparable to the advective ones. This process is called the forward (*i.e.* toward smaller scales) *turbulent cascade.*

- Quasi-periodic and spatially smooth versus broad-band.
  Turbulence develops a broad spectrum in (ω, k) space even when given smooth or periodic initial conditions or forcing. The spectra of atmospheric and oceanic flows can show some indication of preferred time and spatial scales, but have broad-band shape (examples: atmospheric near-surface wind frequency spectrum and oceanic internal wave frequency spectrum). Spectra often have power-law forms over wide frequency and wavenumber ranges (*e.g.* the famous $k^{-5/3}$ form of the Kolmogorov cascade in three-dimensional turbulence and the ω^{-5}

frequency spectrum for surface waves in the open ocean).

- Predictable versus unpredictable. A turbulent flow is predictable for only a limited period of time, usually of the order of several eddy turnover times, L/U. On the other hand statistical properties of the flow can be quite stable, hence in principle predictable. This situation is reminiscent of thermodynamics: if we put two bodies of different temperatures in contact we can predict their final temperature, but not the trajectory of every single molecule.

- Smooth versus sensitive dependence.
  The evolution of the partial differential equations of fluid dynamics is ill-posed in the classical sense, in that small changes in any aspect of the problem formulation (initial conditions, boundary conditions, model parameters, *etc.*) lead to large macroscopic changes in the solution after a finite time, the so-called predictability time.

- Statistically regular versus statistically irregular.
  Some statistical measures of turbulence have nearly Gaussian probability distributions, but the overall structure of turbulent fields is highly non-Gaussian with rare and intense events relatively more probable (i.e. turbulence has bursts).

- Globally ordered versus locally ordered.
  Global ordering is rare for large Re, unless it is established through ordered forcing or domain geometry. But local ordering, particularly of the vorticity field, is nearly universal. This local ordering is the *coherent structures.*

- Persistent versus transient, evanescent, chaotically recurrent patterns.
  The evident ordering in the flow patterns is usually transient. Coherent structures appear and disappear in an unpredictable sequence.

In real turbulent flows neither pair of the antonyms is entirely false. However much of our understanding of turbulence lies in comprehending how flows transitions from one member of the antonym to the other.

1.2 Governing equations

We will take a simplified form of the governing equations for use in this course, *viz.*, the Boussinesq equations. The ocean and the atmosphere have a more complex thermodynamics than these equations do, but this is largely extraneous to the fundamental behaviors of turbulence and thus will be ignored. The three dimensional

Boussinesq system is,

$$\frac{\partial \boldsymbol{u}}{\partial t} + \underbrace{(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}}_{inertia} = -\frac{1}{\rho_0}\nabla p + \underbrace{\nu\nabla^2 \boldsymbol{u}}_{friction} + \left[\underbrace{b\hat{\boldsymbol{z}}}_{buoyancy} - \underbrace{f\hat{\boldsymbol{z}} \times \boldsymbol{u}}_{Coriolis}\right], \tag{1.4}$$

$$\left[\underbrace{\frac{\partial b}{\partial t} + (\boldsymbol{u} \cdot \nabla)b}_{advection} = \underbrace{\kappa\nabla^2 b}_{diffusion}\right], \tag{1.5}$$

$$\left[\underbrace{\frac{\partial c}{\partial t} + (\boldsymbol{u} \cdot \nabla)c}_{advection} = \underbrace{\kappa\nabla^2 c}_{diffusion}\right], \tag{1.6}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1.7}$$

where p is the pressure, f is the Coriolis frequency associated with planetary rotation, and the vertical versor $\hat{\boldsymbol{z}}$ is assumed to be parallel to both gravity and the axis of rotation. The buoyancy b is defined in terms of density as $\rho = \rho_0(1 - g^{-1}b)$. Notice that b has the dimension of an acceleration. These equations must be complemented by forcing, boundary and initial conditions to obtain a well posed problem. We can also consider the evolution of a passive tracer c: it satisfies the same equation as b, but it has no influence on the evolution of $\boldsymbol{u}$. [The Boussinesq equations follow from the full Navier-Stokes equations if one neglects all density fluctuations except those due to heat, salt, and humidity].

These equations have conservative integral invariants for energy, and all powers and other functionals of buoyancy, in the absence of friction and diffusion. For non-conservative dynamics, the energy and scalar variance satisfy the equations,

$$\frac{\partial E}{\partial t} = -\epsilon, \qquad \frac{\partial B}{\partial t} = -\epsilon_B, \tag{1.8}$$

where,

$$[E, B, \epsilon, \epsilon_B] = \int\int\int \mathrm{d}\boldsymbol{x}\left[\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u} - bz, b^2, \nu\,\nabla\boldsymbol{u} : \nabla\boldsymbol{u} + \kappa z\nabla^2 b, \kappa\,\nabla b \cdot \nabla b\right], \tag{1.9}$$

In deriving (1.8), it is assumed that there are no boundary fluxes of energy or scalar variance. These integrals measures of the flow can only decrease with time through the action of molecular viscosity and diffusivity. The only exception is the compressive work, $\kappa z\nabla^2 b$, which can act as a source of mechanical energy. We will return to this issue in the chapter on convection.

Every problem we will consider lies within the set of solutions of the PDE system in (1.4) through (1.7). No general solution is known, nor is any in prospect, because we do not know a mathematical methodology that seems powerful enough. However computers are giving us access to progressively better particular solutions, *i.e.* with progressively larger Re.

The brackets in (1.4) through (1.5) contain the effects of buoyancy and rotation, and these terms are ignored in the classical literature on turbulence, which deals with uniform density fluids in inertial reference frames. In these simpler circumstances, the Boussinesq system is called the incompressible Navier-Stokes equations, or with the further elimination of the frictional term, the incompressible Euler equations.

Because of the lack of general solutions to the Boussinesq equations, it is useful to identify which terms might be neglected in specific situations in order to simplify the problem and make analytical progress. The relative size of the various terms that appear in (1.4) through (1.7) can be estimated in terms of nondimensional numbers. The ones that will be most useful in this class are as follows.

- **Inertia and friction**

  The **Reynolds number** is defined as

  $$Re \equiv \frac{UL}{\nu} \tag{1.10}$$

  Here U and L are characteristic velocity and length scales of the flow and ν is the kinematic viscosity of the fluid. The Reynolds number measures the ratio of inertia and friction,

  $$\frac{|(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}|}{|\nu\nabla^2\boldsymbol{u}|} \approx \frac{U/L\ U}{\nu U/L^2} = Re. \tag{1.11}$$

  Equivalently, the Reynolds number is the ratio of the characteristic scale of the flow L and the scale at which momentum is dissipated $l = \nu/U$. In turbulent flows $Re \gg 1$, advective dominance $\Rightarrow$ nonlinear dynamics $\Rightarrow$ chaotic evolution and broadband spectrum.

  The focus of this course is on turbulence in the Earth's ocean and atmosphere. Typical values for ν near the Earth's surface are 1.5×10^{-5} m^2 s^{-1} for air and 1.0×10^{-6} m^2 s^{-1} for water. These values are small enough, given typical velocities U, that $Re \gg 1$ on all spatial scales L from the finescale of about 1 m to the planetary scale of about 10^4 km. For example, $U = 1$ m s^{-1} and $L = 10^3$ m give $Re = 10^9 - 10^{10}$ respectively for the atmosphere and the ocean.

  For $Re \gg 1$, the frictional term is small, at least at the scale L. Paradoxically, however, the dissipation terms in (1.8) control the energetics of the system. Thus, there must be a profound difference in solutions between the asymptotic tendency as $Re \to \infty$, and the Euler limit, $Re = \infty$ or $\nu = 0$. The difference is that as long as $Re \leq \infty$, there are small scales at which friction becomes important and Re is small.

  It is instructive to check how Re controls the behavior of solutions in a real flow. Let us consider a fluid of uniform density in an inertial reference system, *i.e.* let
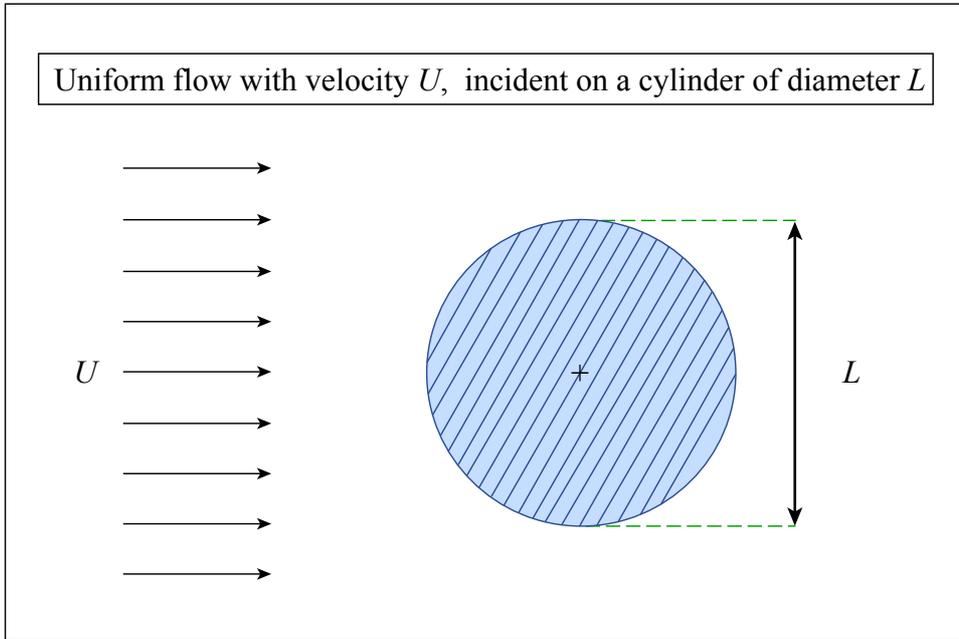
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Uniform flow with velocity U, incident on a cylinder of diameter L

U

L

Figure by MIT OCW.

Figure 1.7: Uniform flow with velocity U, incident on a cylinder of diameter L.

us neglect rotation and variations in buoyancy in (1.4) and (1.5) ($b = f = 0$). A classical example is a uniform flow incident on a cylinder (figure 1.7). Figures 1.8 through 1.10 show how the flow past the cylinder changes for different Reynolds numbers. And the movie by Wim de Leeuw at $Re = 22000$ gives you the feeling of how complex is the flow that develops behind the obstacle (run dns-midi.mpg).

- **Advection and diffusion**

$$Pe \equiv \frac{UL}{\kappa} \tag{1.12}$$

The **Peclet number** is the direct analog of Re for a conserved tracer c with a diffusivity κ and measures the relative importance of advection and diffusion. At large Pe, the tracer evolution is dominated by advection. Once more, the limit $Pe \to \infty$ is very different from $Pe = \infty$, because dissipation, no matter how small, eventually is responsible for removing structure from the tracer field (Fig. 1.11).

- **Friction and diffusion**

$$Pr \equiv \frac{\nu}{\kappa} \tag{1.13}$$

The frictional length scale is $l_\nu = \nu/U$ and the diffusive length scale $l_\kappa = \kappa/U$. The **Prandtl number** is defined as the ratio of these two length scales, $Pr \equiv$

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Figure 1.8: Flow past a cylinder at $R = 0.16$ and $R = 1.54$ (Van Dyke 1982).

l_ν / l_κ. The Prandtl number is a property of the fluid, not of the particular flow. Hence there is a restriction on the transfer of information from experiments with one fluid to those with another. For $Pr > 1$ the scales at which friction becomes important are larger than those for diffusion and, at some small scale, we expect to find smooth velocity fields together with convoluted tracer fields. For $Pr < 1$ we expect the opposite. The Prandtl numbers for air and water are 0.7 and 12.2 respectively. The paper by Paparella and Young (Journal of Fluid Mechanics, 2002) shows examples of flows with low and high Pr.

- **Inertia and Coriolis**

$$Ro \equiv \frac{U}{fL} \tag{1.14}$$

The **Rossby number** Ro measures the relative importance of the real inertial forces and the fictitious Coriolis force, that appear because of the rotating reference system. Thus Ro measures the importance of rotation in the problem at hand. $Ro \gg 1$ characterizes essentially non-rotating turbulence, while $Ro \leq 1$ flows are strongly affected by rotation (Fig. 1.12).

- **Buoyancy and diffusion**

$$Ra \equiv \frac{\Delta b L^3}{\kappa \nu} \tag{1.15}$$

In convective problems, motions are generated by imposing an unstable density stratification on the fluid ($\partial b / \partial z < 0$). In these problems, it is useful to characterize turbulence in terms of the **Rayleigh number**, *i.e.* the ratio of the diffusive timescale $t_\kappa = L^2 / \kappa$ and the buoyancy timescale $t_b = (L/\Delta b)^{1/2}$. The buoyancy scale Δb is the buoyancy difference maintained across the layer depth L through external forcing. If the forcing is imposed by maintaining a

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Citation: Van Dyke, Milton. *An album of fluid motion.* Stanford,
CA: Parabolic Press, 1982, p. 176. ISBN: 9780915760039.

Figure 1.9: Flow past a cylinder at $R = 9.6$, $R = 13.1$, and $R = 26$ (Van Dyke 1982)

Figure 1.10: Wake behind two cylinders at $R = 1800$, and homogeneous turbulence behind a grid at $R = 1500$ (Van Dyke 1982).

temperature difference ΔT, then one has $\Delta b = g\alpha\Delta T$, where α is the coefficient of thermal expansion of the fluid, and g the acceleration of gravity. Convection starts if $t_\kappa \gg t_b$, *i.e.* if $RaPr \gg 1$, when diffusion is too slow to change substantially the buoyancy of water/air parcels as they rise (Fig. 1.13).

- **Buoyancy and inertia**

$$Ri \equiv \frac{\partial b/\partial z}{|\partial \boldsymbol{u}/\partial z|^2} \tag{1.16}$$

In the presence of stable buoyancy stratification, vertical motions tend to be suppressed, but turbulence can still emerge, if there is enough energy in the horizontal velocity field. A useful parameter to characterize the flow in these problems is the ratio of the buoyancy timescale $t_b = (L/\Delta b)^{1/2} = 1/(\partial b/\partial z)^{1/2}$ and the inertial timescale due to horizontal shears in the flow $t_i = L/U = 1/(\partial \boldsymbol{u}/\partial z)$. This ratio is called the gradient Richardson number Ri. If $Ri \ll 1$, buoyancy can be neglected in the momentum equations, and it becomes a passive scalar with no feedbacks on the dynamics (Fig. 1.14).

A final remark about the only term that never appeared explicitly in the nondimensional numbers presented: the pressure force. Pressure can be formally eliminated from the equations. This is a consequence of the Boussinesq approximation. We simply need to take the divergence of the momentum equation in (1.4) and note that $\nabla \cdot \boldsymbol{u}_t = 0$ because of incompressibility. This yields the relation,

$$\nabla^2 p = \rho_0 \nabla \cdot \left[-(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nu\nabla^2\boldsymbol{u} + b\hat{\boldsymbol{z}} - f\hat{\boldsymbol{z}} \times \boldsymbol{u} \right]. \tag{1.17}$$
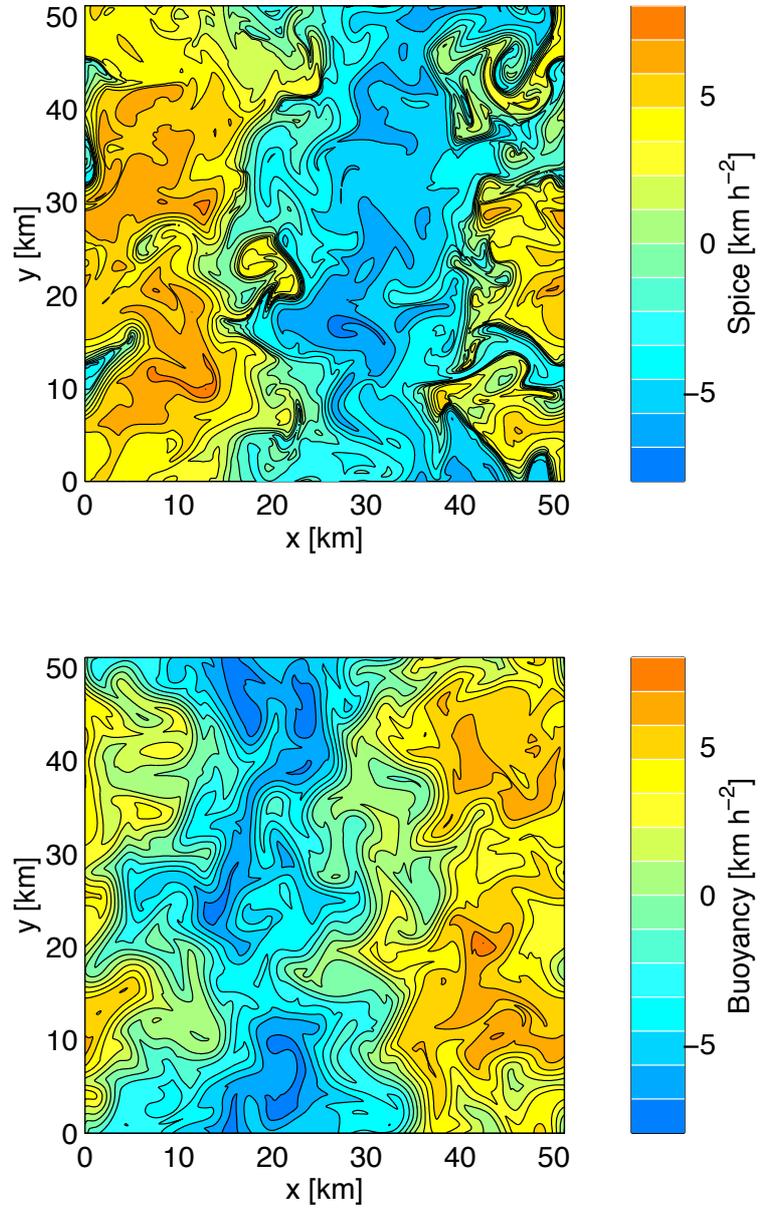
Figure 1.11: Contours of two tracers advected by a two dimensional flow field. The two tracers have $Pe = 100$ (upper panel) and $Pe = 10$ (lower panel).

Figure removed due to copyright restrictions.
Citation: http://www.the-planet-jupiter.com.

Figure 1.12: Rotational effects produce the well-known jets in the atmosphere of Jupiter. The Ro can be estimated to be of order 0.1 using $U = 300$ m/s, $f = 2.5 \cdot 10^{-4}$ s^{-1}, and $L = 10,000 km$).

Figure removed due to copyright restrictions.
Citation: Vincent, Alain, and David A. Yuen, "Statistical Physics, Plasmas, Fluids, and Related Interdisciplinary Topics." *Physical review E,* 2000.

Figure 1.13: Convection in air at a $Ra = 10^{12}$ and $Ra = 10^{14}$ (Alain Vincent and David A. Yuen, Phys. Review E, 2000).

Figure removed due to copyright restrictions.

Figure 1.14: Kelvin Helmholtz instability in the atmosphere. Picture taken near Washington, D.C.. Darker areas are clouds.

Since there are no time derivatives in (1.17), pressure is a purely diagnostic field, which is wholly slaved to $\boldsymbol{u}$. It can be calculated from (1.17) and then substituted for the pressure gradient force in the momentum equations. Its role is to maintain incompressibility under the action of all other forces. It would be redundant to introduce nondimensional parameters involving pressure, because any such parameters can be expressed as combinations of the parameters already discussed.