Particle Dispersion

Random Flight – Lagrangian dispersion

As an example, we examine the random flight model, which assumes that the accelerations have a stochastic component and use Newton's equations

$$\boxed{\begin{aligned} d\mathbf{X} &= \mathbf{V}dt \\ d\mathbf{V} &= \mathbf{A}dt + \beta d\mathbf{R} \end{aligned}}$$

where $\mathbf{A}$ is the acceleration produced by deterministic (or large-scale) forces. We include random accelerations with the random increment $d\mathbf{R}$ satisfying $\langle dR_i\, dR_j \rangle = \delta_{ij} dt$.

As examples, consider a drag law for the acceleration

$$\mathbf{A} = -r(\mathbf{V} - \mathbf{u})$$

with $\mathbf{u}$ being the water velocity. The dispersion is determined by β and r; from the equations, we can show that

$$\begin{aligned} \langle V_i \rangle &\to u_i \\ \langle (V_i - u_i)(V_j - u_j) \rangle &\to \frac{\beta^2}{2r}\delta_{ij} \\ \langle X_i(t) X_j(t) \rangle &\to \langle X_i(0) X_j(0) \rangle + \frac{\beta^2}{r^2}\delta_{ij} t \end{aligned}$$

The latter corresponds to a diffusivity of $\kappa = \beta^2/2r^2$.

- Area grows like $4\kappa t$ ($6\kappa t$ in 3-D)
- Velocity variance is $r\kappa$

`Demos, Page 1: Random flight` *<dispersion>* *<mean sq displacement>*

Taylor dispersion

In 1922, Taylor described the dispersion under the assumption that the Lagrangian velocity had a known covariance structure. He considered just

$$\frac{\partial}{\partial t}\mathbf{X} = \mathbf{V}(t)$$

We find that

$$\frac{\partial}{\partial t}X_iX_j = V_iX_j + X_iV_j$$

and, in the ensemble average,

$$\frac{\partial}{\partial t}\langle X_iX_j\rangle = \langle V_iX_j\rangle + \langle X_iV_j\rangle$$

If we substitute

$$\mathbf{X} = \mathbf{X}_0 + \int_0^t \mathbf{V}(t')dt'$$

and look at the case where $\langle \mathbf{V}\rangle = 0$ and the flow is stationary, we have

$$\frac{\partial}{\partial t}\langle X_iX_j\rangle = \int_0^t dt'\ R_{ij}^L(t') + R_{ji}^L(t')$$

where R_{ij}^L is the covariance of the Lagrangian velocities

$$R_{ij}^L(t) = \langle V_i(t_0+t)V_j(t_0)\rangle$$

For isotropic motions $R_{ij}^L(t) = U^2R^L(t)\delta_{ij}$ with $R^L(t)$ being the autocorrelation function; the change in x-variance is given by

$$\frac{\partial}{\partial t}\langle X^2\rangle = 2U^2\int_0^t R^L(t)$$

From this formula, we see that

- For short times,

$$\langle X^2\rangle = U^2t^2$$

- For long times, if the integral $T_{int} = \int_0^\infty R^L(t)dt$ is finite and non-zero,

$$\langle X^2\rangle = 2U^2T_{int}t$$

Relation to diffusivity

Consider the diffusion of a passive scalar

$$\frac{\partial}{\partial t}C = -\nabla \cdot [\mathbf{u} - \kappa\nabla]C$$

and define moments of the distribution

$$\langle x^n \rangle = \frac{\int x^n C}{\int C}$$

Integrating the diffusion equation gives conservation of the total scalar, under the assumption that the initial distribution is compact and the values decay rapidly at infinity

$$\frac{\partial}{\partial t}\int C = \oint \hat{\mathbf{n}} \cdot [\kappa\nabla C - \mathbf{u}C] = 0$$

The first moment gives

$$\frac{\partial}{\partial t}\int xC = \int \nabla \cdot [x(\kappa\nabla C - \mathbf{u}C] + \int uC - \kappa\frac{\partial}{\partial x}C = \int uC - \nabla \cdot \kappa C\hat{\mathbf{x}} + \frac{\partial\kappa}{\partial x}C = \int (u + \frac{\partial\kappa}{\partial x})C$$

In the absence of flow and with a constant κ, $\frac{\partial}{\partial t}\langle \mathbf{x} \rangle = 0$. Otherwise, the center of mass migrates according to a weighted version of $\mathbf{u} + \nabla\kappa$: it moves with the flow and upgradient in diffusivity.

The second moment

$$\frac{\partial}{\partial t}\langle x^2 \rangle = 2\langle xu \rangle + 2\langle \frac{\partial}{\partial x}(x\kappa) \rangle$$

implies that

$$\frac{\partial}{\partial t}\left[\langle x^2 \rangle - \langle x \rangle^2\right] = 2\left[\langle xu \rangle - \langle x \rangle\langle u \rangle + \langle x\frac{\partial\kappa}{\partial x} \rangle - \langle x \rangle\langle \frac{\partial\kappa}{\partial x} \rangle\right] + 2\langle \kappa \rangle$$

For uniform flow and constant diffusivity, the blob spreads in x at a rate 2κ. Thus we can identify the effective diffusivity

$$\kappa = U^2 T_{int}$$

Strain in the flow and curvature in κ will alter the rate of spread.

Small amplitude motions

If we assume that the scale of a typical particle excursion over time T_{int} is small compared to the scale over which the flow varies, we can relate the Lagrangian and Eulerian statistics. The displacement $\xi_i = X_i(t) - X_i(0)$ satisfies

$$\frac{\partial}{\partial t}\xi_i = u_i(\mathbf{x}+\boldsymbol{\xi}, t) \simeq u_i(\mathbf{x}, t) + \xi_j \frac{\partial}{\partial x_j} u_i(\mathbf{x}, t) + \dots$$

and we can substitute the lowest order solution

$$\xi_i(t) = \int_0^t dt' u_i(\mathbf{x}, t')$$

into the second term above to write

$$\frac{\partial}{\partial t}\xi_i = u_i(\mathbf{x}, t) + \frac{\partial}{\partial x_j}\int_0^t u_j(\mathbf{x}, t') u_i(\mathbf{x}, t)$$

and average, recognizing that the mean Lagrangian velocity is just $\langle \frac{\partial}{\partial t}\xi_i \rangle$:

$$\langle u_i^L \rangle = \langle u_i \rangle + \frac{\partial}{\partial x_j}\int_0^t \langle u_i(\mathbf{x}, t) u_j(\mathbf{x}, t') \rangle$$

For simplicity, we assume that the turbulent velocities are large compared to the mean; then this becomes

$$\langle u_i^L \rangle = \langle u_i \rangle + \frac{\partial}{\partial x_j}\int_0^t R_{ij}(\mathbf{x}, t-t') = \langle u_i \rangle + \frac{\partial}{\partial x_j}\int_0^t d\tau R_{ij}(\mathbf{x}, \tau)$$

Let us assume that the integrals with respect to τ exist and split the covariance into its symmetric and antisymmetric parts

$$\langle u_i^L \rangle = \langle u_i \rangle + \frac{\partial}{\partial x_j} D_{ij}^s(\mathbf{x}) + \frac{\partial}{\partial x_j} D_{ij}^a$$

with

$$K_{ij} \equiv D_{ij}^s = \frac{1}{2}\int_0^\infty R_{ij}(\mathbf{x}, \tau) + R_{ji}(\mathbf{x}, \tau) \quad , \quad D_{ij}^a = \frac{1}{2}\int_0^\infty R_{ij}(\mathbf{x}, \tau) - R_{ji}(\mathbf{x}, \tau)$$

We can write an arbitrary antisymmetric tensor in terms of the unit antisymmetric tensor

$$D_{ij}^a = -\epsilon_{ijk}\Psi_k$$

so that the contribution to the Lagrangian velocity is

$$u_i^S = -\epsilon_{ijk}\frac{\partial}{\partial x_j}\Psi_k \quad , \quad \mathbf{u}^S = -\nabla \times \Psi$$

Note that the antisymmetric part of the contribution to the Lagrangian velocity is nondivergent:

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}D^a_{ij}(\mathbf{x}) = \nabla\cdot\mathbf{u}^S = 0$$

Thus the Lagrangian mean velocity has contributions from the mean Eulerian flow, from the Stokes' drift, and a term which tends to move into regions of higher diffusivity

$$\langle u^L_i\rangle = \langle u_i\rangle + u^S_i + \frac{\partial}{\partial x_j}K_{ij}(\mathbf{x})$$

We will discuss the meanings of these terms in more detail next.

Random Rossby Waves

Consider a randomly-forced Rossby wave in a channel:

$$\frac{\partial}{\partial t}\nabla^2\psi + J(\psi, \nabla^2\psi + y) = \frac{U_0}{\ell}\gamma\mathrm{Re}[r(t)e^{\imath kx}]\sin(\ell y) - \gamma\nabla^2\psi$$

where r is randomly distributed on a disk of radius r_0. This gives a streamfunction

$$\psi = \frac{U_0}{\ell}\mathrm{Re}[a(t)e^{\imath kx}]\sin(\ell y)$$

with

$$\frac{d}{dt}a + (\gamma + \imath\,\omega)a = \frac{\omega\gamma}{\beta k}r$$

and $\omega = -\beta k/(k^2+\ell^2)$.

$$a = \frac{\gamma}{2}\int_{-\infty}^{t} d\tau e^{-(\gamma-\frac{1}{2}\imath)\tau}r(t-\tau)$$

Stokes' drift

Consider first the steady wave case.

$$\psi = \frac{\epsilon}{\pi}\sin(\pi[x-t])\sin(\pi y)$$

We look at the particle trajectories by solving the Lagrangian equations as above

$$\frac{\partial}{\partial t}\boldsymbol{\xi} = \mathbf{u}(\mathbf{x}+\boldsymbol{\xi}, t)$$

For small ϵ (which is the ratio of the flow speed to the phase speed, we can find an approximate solution (as before) by iterating

$$\begin{aligned}\frac{\partial}{\partial t}\xi_i &\simeq u_i(\mathbf{x}, t') + \xi_i \frac{\partial}{\partial x_j} u_i(\mathbf{x}, t) + \dots \\ &\simeq u_i(\mathbf{x}, t) + \frac{\partial}{\partial x_j}\int_0^t u_j(\mathbf{x}, t') u_i(\mathbf{x}, t) dt'\end{aligned}$$

The mean Lagrangian drift is therefore

$$\overline{\frac{\partial}{\partial t}}\xi_i = \frac{\partial}{\partial x_j}\int_0^t R_{ij}(\mathbf{x}, \tau) d\tau$$

Treating the mean as a phase average gives

$$R_{ij}(\tau) = \frac{\epsilon^2}{2}\begin{pmatrix} \cos\pi\tau\cos^2\pi y & \sin\pi\tau\sin\pi y\cos\pi y \\ -\sin\pi\tau\sin\pi y\cos\pi y & \cos\pi\tau\sin^2\pi y \end{pmatrix}$$

the integral gives

$$D_{ij}(t) = \int_0^t R_{ij}(\tau) d\tau = \frac{\epsilon^2}{2\pi}\begin{pmatrix} \sin\pi t\cos^2\pi y & (1-\cos\pi t)\sin\pi y\cos\pi y \\ -(1-\cos\pi t)\sin\pi y\cos\pi y & \sin\pi\tau\sin^2\pi y \end{pmatrix}$$

so that the drift is

$$\begin{aligned} u_L &= \overline{\frac{\partial}{\partial t}}\xi_1 = \frac{\epsilon^2}{2}\cos(2\pi y)[1-\cos(\pi t)] \\ v_L &= \overline{\frac{\partial}{\partial t}}\xi_2 = \frac{\epsilon^2}{2}\sin(2\pi y)\sin\pi t \end{aligned}$$

Note that there is a time-averaged drift

$$\overline{u_L} = \frac{\epsilon^2}{2}\cos(2\pi y)$$

prograde on the walls and retrograde in the center.

Note that we can split D_{ij} as usual:

$$\begin{aligned} u_i^L &= \frac{\partial}{\partial x_j} K_{ij} + \frac{\partial}{\partial x_j} D_{ij}^a \\ &= \frac{\partial}{\partial x_j} K_{ij} - \frac{\partial}{\partial x_j} \epsilon_{ijk} \Psi_k \\ &= \frac{\partial}{\partial x_j} K_{ij} + u_i^S \end{aligned}$$

with the first term giving the up-diffusive-gradient transport associated with the symmetric part of $\int R_{ij}$ and the second, nondivergent part, arising from the antisymmetric term, gives the Stokes drift. For the primary wave,

$$K_{ij} = \frac{\epsilon^2}{2\pi} \begin{pmatrix} \sin \pi t \cos^2 \pi y & 0 \\ 0 & \sin \pi t \sin^2 \pi y \end{pmatrix}$$

and has no time average, while

$$\Psi_3 = -\frac{\epsilon^2}{4\pi}(1 - \cos \pi t) \sin 2\pi y$$

produces the nondivergent Stokes drift (and does have a mean). `Demos, Page 7: drift` *`<amp=0.2>`* *`<amp=0.2 comoving>`* *`<amp=1.0>`* *`<amp=1.0 comoving>`* *`<stokes drift>`* *`<mean>`*

FINITE AMPLITUDE

In the frame of reference of the wave ($\mathbf{X}' = \mathbf{X} - \mathbf{c}t$)

$$\frac{\partial}{\partial t} \mathbf{X}' = \mathbf{u}(\mathbf{X}') - \mathbf{c} = \hat{\mathbf{z}} \times \nabla(\psi + cy)$$

Thus particles simply move along the streamlines. At some Lagrangian period T_L, the particle will have moved one period to the left so that

$$X'(T_L) = X(0) - \lambda = X(T_L) - cT_L \quad \Rightarrow \quad u_L = \frac{X(T_L) - X(0)}{T_L} = c - \frac{\lambda}{T_L} = c\,(1 - \frac{T_E}{T_L})$$

Stokes drifts occur when the Lagrangian period differs from the Eulerian period. Trapped particles have

$$X'(T_L) = X(0) = X(T_L) - cT_L \quad \Rightarrow \quad u_L = \frac{X(T_L) - X(0)}{T_L} = c$$

Back to random wave

From

$$\psi = \frac{U_0}{\ell}\mathrm{Re}[a(t)e^{\imath kx}]\sin(\ell y)$$

with

$$a = \frac{\gamma}{2}\int_{-\infty}^{t} d\tau e^{-(\gamma-\frac{1}{2}\imath)\tau} r(t-\tau)$$

we find

$$\overline{\psi(x,y,t)\psi(x',y',t')} =$$

$$\frac{U_0^2}{2\ell^2}e^{-\gamma(t-t')}\cos[k(x-x')-\omega(t-t')]\sin(\ell y)\sin(\ell y')$$

$$R_{mn}(\tau) = \frac{1}{2}U_0^2 e^{-\gamma\tau}\begin{pmatrix} \cos\omega\tau\cos^2\ell y & \frac{k}{\ell}\sin\omega\tau\sin\ell y\cos\ell y \\ -\frac{k}{\ell}\sin\omega\tau\sin\ell y\cos\ell y & \frac{k^2}{\ell^2}\cos\omega\tau\sin^2\ell y \end{pmatrix}$$

This gives

$$D_{mn} = \frac{1}{2}\frac{U_0^2}{\gamma^2+\omega^2}\begin{pmatrix} \gamma\cos^2\ell y & \omega\frac{k}{\ell}\sin\ell y\cos\ell y \\ -\omega\frac{k}{\ell}\sin\ell y\cos\ell y & \gamma\frac{k^2}{\ell^2}\sin^2\ell y \end{pmatrix}$$

The diffusivities and Stokes' drift are given by

$$(K_{11},\ K_{22}) = \frac{1}{2}U_0^2\frac{\gamma}{\gamma^2+\omega^2}\left(\cos^2\ell y,\ \frac{k^2}{\ell^2}\sin^2\ell y\right)$$

$$\Psi_3 = -A_{12} = A_{21} = -\frac{1}{2}U_0^2\frac{k}{\ell}\frac{\omega}{\gamma^2+\omega^2}\sin\ell y\cos\ell y$$

$$u^L = u^S = \frac{1}{2}U_0^2 k\frac{\omega}{\gamma^2+\omega^2}\cos 2\ell y$$

$$v^L = \frac{1}{2}U_0^2\frac{k^2}{\ell}\frac{\gamma}{\gamma^2+\omega^2}\sin 2\ell y$$

`Demos, Page 8: structure` *<K,u,v>* `Demos, Page 8: stokes drift` *<lin vs act sd>* *<mean drift>*

Conclusions:

- Rossby waves cause mean westward drifts at the edges and eastward drifts in the center.
- Eddy diffusivities are spatially variable and anisotropic.

Chaotic advection

We start with the basic wave

$$\psi = \frac{\epsilon}{\pi} \sin(\pi[x-t])\sin(\pi y)$$

and add a small amount of a second wave

$$\psi = \sqrt{1-16\alpha^2}\frac{\epsilon}{\pi} \sin(\pi[x-t])\sin(\pi y) + \alpha\frac{\epsilon}{\pi}\sin(4\pi[x-c_1 t])\sin(4\pi y)$$

`Demos, Page 8: psi` <*alpha=0*> <*alpha=0.01*> <*alpha=0.1*>

When we have α non-zero, the trajectories become less regular in the vicinity of the stagnation points. A line of particles approaching the point begins to fold, with some fluid crossing into the interior and some being ejected. Which way a parcel goes depends on the phase of the perturbing wave as it nears the stagnation point.

`Demos, Page 9: lobe dynamics` <*alpha 0.008*>

We can look at Poincaré sections (snapshots at the period of the perturbing wave) at various amplitudes to see the mixing regions `Demos, Page 9: poincare sections` <*alpha=0*> <*alpha=0.002*> <*alpha=0.004*> <*alpha=0.008*> <*alpha=0.016*> <*alpha=0.032*> <*alpha=0.064*> <*alpha=0.128*>

The mixing across the channel is still blocked for α small enough < 0.05 so the mixing is still diffusion-limited, although some gain is realized by enhanced flux out of the wall and a decrease in the width of the blocked region.

`Demos, Page 9: Continuum` <*steady*> <*weak*> <*strong*>

References

Flierl, G.R. (1981) Particle motions in large amplitude wave fields. *Geophys. Astrophys. Fluid Dyn.*, **18**, 39-74.

Pierrehumbert, R.T. (1991) Chaotic mixing of tracer and vorticity by modulated travelling Rossby waves. *Geophys. Astrophys. Fluid Dyn.*, **58**, 285-319.

Active Tracers

We review mixing length theory applied to a set of active scalars (think in terms of biological properties):

$$\frac{D}{Dt}b_i + \nabla\cdot(\mathbf{u}_{bio}b_i) - \nabla\kappa\nabla b_i = \mathcal{B}_i(\mathbf{b},\mathbf{x},t)$$

Split the field into an eddy part which varies rapidly in space and time and a mean part which changes over larger (order $1/\epsilon$) horizontal distances and longer (order $1/\epsilon^2$) times:

$$b_i = \overline{b}_i(z|\mathbf{X},T) + b_i'(\mathbf{x},z,t|\mathbf{X},T)$$

We must allow for short vertical scales in both means and fluctuations. Counterbalancing this difficulty is the fact that vertical velocities tend to be weak (order $\frac{U}{fL}\times\frac{UH}{L}$). We assume the mean flows are small $\overline{u} \sim \varepsilon \mathbf{u}'$ and the coefficients in the reaction terms vary rapidly in the vertical but slowly horizontally and in time.

$$\frac{D}{Dt} - \nabla \cdot \kappa \nabla \longrightarrow$$

$$\left[\frac{\partial}{\partial t} + u'_m \nabla_m - \nabla \cdot \kappa \nabla - \frac{\partial}{\partial z} \kappa_v \frac{\partial}{\partial z}\right]$$

$$+\varepsilon \left[u'_m \ \boldsymbol{\nabla}_m + \overline{u}_m \nabla_m - \boldsymbol{\nabla}_m \kappa_{mn} \nabla_n - \nabla_m \kappa_{mn} \boldsymbol{\nabla}_n + \frac{Ro}{\varepsilon} w' \frac{\partial}{\partial z}\right]$$

$$+ \varepsilon^2 \left[\frac{\partial}{\partial T} + \overline{u}_m \boldsymbol{\nabla}_m - \boldsymbol{\nabla}_m \kappa_{mn} \boldsymbol{\nabla}_n + \frac{Ro}{\varepsilon} \overline{w} \frac{\partial}{\partial z}\right]$$

Vertical Structure

1) We assume the case with no flow has a *stable* solution:

$$\frac{\partial}{\partial z} w_{bio} \overline{b}_i = \frac{\partial}{\partial z} \kappa_v \frac{\partial}{\partial z} \overline{b}_i + \mathcal{B}_i(\overline{\mathbf{b}}, z | \mathbf{X}, T)$$

`Demos, Page 10:  bio dynamics` <*growth rates*>

2) The eddy-induced perturbations satisfy

$$\left[\frac{\partial}{\partial t} + \mathbf{u}' \cdot \nabla - \nabla \cdot \kappa \nabla\right] b'_i + \frac{\partial}{\partial z} w_{bio} b' =$$

$$\sum_j \frac{\partial \mathcal{B}_i}{\partial b_j} b'_j - \mathbf{u}' \cdot \boldsymbol{\nabla} \overline{b}_i \equiv \mathcal{B}_{ij} b'_j - \mathbf{u}' \cdot \boldsymbol{\nabla} \overline{b}_i$$

with $\boldsymbol{\nabla} = (\partial/\partial X,\ \partial/\partial Y,\ \partial/\partial z)$.

3) The equation for the mean is

$$\left[\frac{\partial}{\partial T} + \overline{\mathbf{u}} \cdot \boldsymbol{\nabla} - \boldsymbol{\nabla} \kappa \boldsymbol{\nabla}\right] \overline{b}_i + \boldsymbol{\nabla} \cdot (\overline{\mathbf{u}' b'}) + \frac{\partial}{\partial z} w_{bio} \overline{b}_i =$$

$$\mathcal{B}_i(\overline{\mathbf{b}}, z | \mathbf{X}, T) + \frac{1}{2} \frac{\partial^2 \mathcal{B}_i}{\partial b_j \partial b_k} \overline{b'_j b'_k}$$

Summary:

Eddies generate fluctuations by horizontal and vertical advection of large-scale gradients, but the strength and structure depends on the biologically-induced perturbation decay rates.

Perturbations generate eddy fluxes and alter the average values of the nonlinear biological terms.

NPZ

A simple biological model (mixed layer):

$$\frac{D}{Dt}P = \frac{\mu PN}{N+k_s} - \frac{g}{\nu}Z[1-\exp(-\nu P)] - d_P P + \nabla\kappa\nabla P$$
$$\frac{D}{Dt}Z = \frac{ag}{\nu}Z[1-\exp(-\nu P)] - d_Z Z + \nabla\kappa\nabla Z$$
$$\frac{D}{Dt}N = -\frac{\mu PN}{N+k_s} + \frac{(1-a)g}{\nu}Z[1-\exp(-\nu P)]$$
$$+ d_P P + d_Z Z + \nabla\kappa\nabla N$$
$$or \quad N = N_T - P - Z$$

Mean-field approach

We can get a very similar picture using the mean-field approximation: take

$$\frac{\partial}{\partial t}\overline{b}_i + \overline{u}\cdot\nabla\overline{b}_i + \nabla\cdot(\overline{\mathbf{u}'b'}) + \frac{\partial}{\partial z}w_{bio}\overline{b}_i - \nabla\kappa\nabla\overline{b}_i = \overline{\mathcal{B}_i(\overline{\mathbf{b}}+\mathbf{b}',\mathbf{x},t)}$$

$$\simeq \mathcal{B}_i(\overline{\mathbf{b}},z|\mathbf{x},t) + \frac{1}{2}\frac{\partial^2\mathcal{B}_i}{\partial b_j\partial b_k}\overline{b_j'b_k'}$$

$$\frac{\partial}{\partial t}\mathbf{b}_i' + \overline{u}\cdot\nabla\mathbf{b}_i' + \nabla\cdot(\mathbf{u}'b_i' - \overline{\mathbf{u}'b'}) + \frac{\partial}{\partial z}w_{bio}\mathbf{b}_i' - \nabla\kappa\nabla\mathbf{b}_i' =$$

$$-\mathbf{u}'\cdot\nabla\overline{b}_i + \mathcal{B}_i(\overline{\mathbf{b}}+\mathbf{b}',\mathbf{x},t) - \overline{\mathcal{B}_i(\overline{\mathbf{b}}+\mathbf{b}',\mathbf{x},t)}$$

or (dropping the quadratic and higher terms)

$$\frac{\partial}{\partial t}\mathbf{b}_i' + \overline{u}\cdot\nabla\mathbf{b}_i' + \frac{\partial}{\partial z}w_{bio}\mathbf{b}_i' - \nabla\kappa\nabla\mathbf{b}_i' \simeq -\mathbf{u}'\cdot\nabla\overline{b}_i + \mathcal{B}_{ij}b_j'$$

The differences are subtle: the MFA does not presume that the scale of $\overline{b}_i$ is large but linearizes in a way which may not be consistent.

Separable Problems

The mesoscale eddy field has horizontal velocities in the near-surface layer which are nearly independent of z, and the vertical velocity increases linearly with depth $w' = s(\mathbf{x}, t)z$. The stretching satisfies

$$s(\mathbf{x}, t) = -\nabla \cdot \mathbf{u}(\mathbf{x}, t)$$

For linear (or linearized perturbation) problems in the near-surface layers, we can separate the physics and the biology using Greens' functions.

We define the Greens function for the horizontal flow problem:

$$\left(\frac{\partial}{\partial t} + \mathbf{u}(\mathbf{x}, t) \cdot \nabla - \nabla \kappa \nabla\right) G(\mathbf{x}, \mathbf{x}', t - t') = \delta(\mathbf{x} - \mathbf{x}')\delta(t - t')$$

The perturbation equations can now be solved:

$$\begin{aligned} b_i' = &- \int d\mathbf{x}' \int dt' G(\mathbf{x}, t|\mathbf{x}', t') u_m'(\mathbf{x}', t') \phi_{m,i}(z, t - t') \\ &- \int d\mathbf{x}' \int dt' G(\mathbf{x}, t|\mathbf{x}', t') s'(\mathbf{x}', t') \varphi_i(z, t - t') \end{aligned}$$

The two functions representing the biological dynamics both satisfy

$$\frac{\partial}{\partial \tau}\varphi_i = \frac{\partial}{\partial z}\kappa_v \frac{\partial}{\partial z}\varphi_i + \mathcal{B}_{ij}\varphi_j$$

with $\mathcal{B}_{ij} = \partial \mathcal{B}_i / \partial b_j$. These give the diffusive/ biological decay of standardized initial perturbations

$$\phi_{m,i}(z, 0) = \boldsymbol{\nabla}_m \overline{b}_i \quad , \quad \varphi_i(z, 0) = z\frac{\partial}{\partial z}\overline{b}_i$$

Simple Example

If we ignore vertical diffusion and advection and consider only one component with $\mathcal{B}_{11} = -\lambda$, we have

$$\phi_{m,i} = e^{-\lambda\tau} \nabla_m \bar{b}_i$$

so that

$$b'_i = -\left[\int d\mathbf{x}' \int dt' e^{-\lambda(t-t')} G(\mathbf{x}, t|\mathbf{x}', t') u'_n(\mathbf{x}', t') \right] \nabla_n \bar{b}_i$$

The eddy flux takes the form

$$\begin{aligned} \overline{u'_m b'} &= -\left[\int d\mathbf{x}' \int dt' e^{-\lambda(t-t')} \overline{u'_m(\mathbf{x}, t) G(\mathbf{x}, t|\mathbf{x}', t') u'_n(\mathbf{x}', t')} \right] \nabla_n \bar{b}_i \\ &= -\left[\int d\mathbf{x}' \int dt' e^{-\lambda(t-t')} R_{mn}(\mathbf{x}, t|\mathbf{x}', t') \right] \nabla_n \bar{b}_i \end{aligned}$$

If we split the right-hand side into symmetric and antisymmetric parts, we find

$$\begin{aligned} \overline{u'_m b'} &= -K^\lambda_{mn} \nabla_n \bar{b}_i + \epsilon_{mnk} \Psi^\lambda_k \nabla_n \bar{b}_i \\ &= -K^\lambda_{mn} \nabla \bar{b}_i - (\epsilon_{mnk} \nabla_n \Psi^\lambda_k) \bar{b}_i + \epsilon_{mnk} \nabla_n (\Psi^\lambda_k \bar{b}_i) \end{aligned}$$

The last term has no divergence and can be dropped. Thus the eddy flux is a mix of diffusion and Stokes' drift:

$$\overline{u'_m b'} = -K^\lambda_{mn} \nabla \bar{b}_i + V^\lambda_m \bar{b}_i$$

Both coefficients depend on the biological time scale λ^{-1}.

For the random Rossby wave case, the Stokes drift term is

$$\mathbf{V}^\lambda = \frac{KE}{(\gamma + \lambda)^2 + \frac{1}{4}} \big(-\cos(2y) \ , \ \ 0 \big)$$

while the diffusivity tensor is

$$K^\lambda_{ij} = 2(\gamma + \lambda) \frac{KE}{(\gamma + \lambda)^2 + \frac{1}{4}} \begin{pmatrix} \cos^2(y) & 0 \\ 0 & \sin^2(y) \end{pmatrix}$$

`Demos, Page 13:  effective coeff` <*effective k,v*>

Not so simple example

"Mixing length" models

$$Flux(b) = -\kappa_e \nabla b$$

even if appropriate for passive tracers are not suitable for biological properties whose time scales may be comparable to those in the physics. Instead, we find

$$\overline{\mathbf{u}'_m b'_i} = -\left[\int d\tau e^{\mathcal{B}_{ij}\tau} R_{mn}(\tau)\right] \nabla_n \overline{b}_j$$

where R_{mn} is the equivalent of Taylor's Lagrangian covariance (but including κ).

We divide the coefficient into symmetric (K) and antisymmetric terms related to the Stokes drift (V)

$$\overline{\mathbf{u}'_m b'_i} = -K^{ij}_{mn} \nabla_n \overline{b}_j + V^{ij}_m \overline{b}_j$$

Note that

- Eddy diffusivities and wave drifts mix different components (flux of P depends on gradient of Z).
- If R has a negative lobe, the biological diffusivities can be larger than that of a passive scalar
- The quasi-equilibrium approximation

$$\mathcal{B}_{ij} b'_j = \mathbf{u}' \cdot \nabla \overline{b}_i$$

works reasonably well in the upper water column. In particular

$$\mathcal{B}_{21} = g\overline{Z}\exp(-\nu\overline{P}) > 0 \quad . \quad \mathcal{B}_{22} = 0$$

so that

$$P' = \frac{1}{\mathcal{B}_{21}} \mathbf{u}' \cdot \nabla \overline{Z} \quad \text{unlike} \quad C' = -\boldsymbol{\xi} \cdot \nabla \overline{C}$$

`Demos, Page 14: complex diffusion` <*`transport coeff: display -geometry +0+0 -bordercolor white -border 20x20 -rotate 90 ~glenn/12.822t/graphics/t0.ps`*> up Z grad flux of Pt1.ps <*`quasiequilibrium fluxes: display -geometry +0+0 -bordercolor white -border 20x20 -rotate 90 ~glenn/12.822t/graphics/t1a.ps`*> down-gradient Kpp,KZZt2.ps

Eulerian-Lagrangian

If $\kappa = 0$, we can relate the relevant form of the Eulerian covariance

$$R_{mn}(\mathbf{x}, t|\mathbf{x}', t') = \overline{u'_m(\mathbf{x}, t) G(\mathbf{x}, t|\mathbf{x}', t') u_n(\mathbf{x}', t')}$$

to Taylor's form. The Greens' function equation

$$\frac{\partial}{\partial t} G + \mathbf{u}(\mathbf{x}, t) \cdot \nabla G = \delta(\mathbf{x} - \mathbf{x}')\delta(t - t')$$

has a solution

$$G(\mathbf{x}, t|\mathbf{x}', t') = \delta\left(\mathbf{x} - \mathbf{X}(t|\mathbf{x}', t')\right)$$

where

$$\frac{\partial}{\partial t} \mathbf{X}(t|\mathbf{x}', t') = \mathbf{u}(\mathbf{X}, t) \quad , \quad \mathbf{X}(t'|\mathbf{x}', t') = \mathbf{x}'$$

gives the Lagrangian position of the particle initially at $\mathbf{x}'$ at time t'. But it is more convenient to back up along the trajectory and let

$$G(\mathbf{x}, t|\mathbf{x}', t') = \delta(\mathbf{x}' - \boldsymbol{\xi}(t - t'|\mathbf{x}, t))$$

where the particle at $\boldsymbol{\xi}$ at time t' passes $\mathbf{x}$ at time t (and takes a time τ for this tranistion). Thus the $\boldsymbol{\xi}$'s give the starting position, which, for stochastic flows varies from realization to realization. We can solve

$$\frac{\partial}{\partial \tau} \boldsymbol{\xi}(\tau|\mathbf{x}, t) = -\mathbf{u}(\boldsymbol{\xi}(\tau|\mathbf{x}, t), t - \tau) \quad , \quad \boldsymbol{\xi}(0|\mathbf{x}, t) = \mathbf{x}$$

for $\tau = 0$ to $\tau = t - t'$ to find $\boldsymbol{\xi}$.

We can now define the generalization of the Lagrangian correlation function used by Taylor

$$\begin{aligned} R_{mn}(t - t', \mathbf{x}) &= \int d\mathbf{x}' \overline{u'(\mathbf{x}, t) G(\mathbf{x}, t|\mathbf{x}', t') u'(\mathbf{x}', t')} \\ &= \overline{u'_m(\mathbf{x}, t) u'_n(\boldsymbol{\xi}(t - t'|\mathbf{x}, t), t - (t - t'))} \end{aligned}$$

or

$$R_{mn}(\tau, \mathbf{x}) = \overline{u'_m(\mathbf{x}, t) u'_n(\boldsymbol{\xi}(\tau|\mathbf{x}, t), t - \tau)}$$

For homogeneous, stationary turbulence (on the scales intermediate between the eddies and the mean), this will be equivalent to Taylor's

$$R_{mn}(\tau) = \overline{u'_m(\mathbf{X}(t' + \tau|\mathbf{x}', t'), t' + \tau) u'_n(\mathbf{x}', t')}$$

but we include inhomogeneity and (for, general G, diffusion).