4. Spectral Estimation from ARMA Forms

Suppose that one has determined the ARMA form (3.6). Then we have

$$\hat{x}(z) = \frac{\hat{\theta}(z)\,\hat{b}(z)}{\hat{a}(z)} \tag{4.1}$$

or setting $z = \exp(-2\pi i s)$,

$$< \hat{x}(\exp(-2\pi i s))\hat{x}(\exp(-2\pi i s))^* >= \Phi(s) = \frac{\left|\hat{b}(\exp(-2\pi i s))\right|^2}{\left|\hat{a}(\exp(-2\pi i s))\right|^2}\sigma_\theta^2. \tag{4.2}$$

If a, b are short sequences, then the calculation in (4.2) of the power density spectrum of the time series can be done essentially analytically. In particular, if $\hat{b} = 1$, so that one has a pure AR, the result is called the "all-pole" method, the power density spectrum being completely determined by the positions of the zeros of $\hat{a}(z)$ in the complex z plane. Under some circumstances, e.g., when the time series is made up of two pure frequencies differing in frequency by Δs in the presence of a white noise background, separation of the two lines can be achieved even if the record length is such that $\Delta s < 1/T$ that is, in violation of the Rayleigh criterion. This possibility and related considerations lead to what is commonly known as maximum entropy spectral estimation.

Exercise. Let $x_m = \sin(2\pi s_1 m) + \sin(2\pi s_2 m) + \theta_m, m = 0, 1, ...N$. Find an AR representation of x_m and use it to calculate the corresponding power density spectrum.

A considerable vogue developed at one time involving use of "exotic" methods of spectral representation, including, especially the maximum entropy method. Over time, the fashion has nearly disappeared because the more astute users recognized that maximum entropy etc. methods are dangerous: they can give seemingly precise and powerful results apparently unavailable in the Fourier methods. But these results are powerful precisely because they rely upon the accuracy of the AR or ARMA etc. model. The sensitivity of e.g., (4.2) to the zero positions in $\hat{a}(z)$ means that if the pure pole representation is not the correct one, the appearance of spectral peaks may be spurious. The exotic methods began to fade with the realization that many apparent peaks in spectra were the result of an incorrect model. Tukey (1984) and others, have characterized ARMA-based methods as "covert", meaning that they hide a whole series of assumptions, and recommend reliance instead on the "overt" or non-parametric Fourier methods which are robust and hide nothing. This is good advice except for individuals who know exactly what they are doing. (Percival and Walden discuss these various methods at length.)