18. Cross-Spectra and Coherence

Definitions

"Coherence" is a measure of the degree of relationship, as a function of frequency, between two time series, x_m, y_m. The concept can be motivated in a number of ways. One quite general form is to postulate a convolution relationship,

$$y_m = \sum_{-\infty}^{\infty} a_k x_{m-k} + n_m \tag{18.1}$$

where the residual, or noise, n_m, is uncorrelated with x_m, $< n_m x_p >= 0$. The a_k are not random (they are "deterministic"). The infinite limits are again simply convenient. Taking the Fourier or $z-$transform of (18.1), we have

$$\hat{y} = \hat{a}\hat{x} + \hat{n}. \tag{18.2}$$

Multiply both sides of this last equation by $\hat{x}^*$ and take the expected values

$$< \hat{y}\hat{x}^* >= \hat{a} < \hat{x}\hat{x}^* > + < \hat{n}\hat{x}^* > \tag{18.3}$$

where $< \hat{n}\hat{x}^* >= 0$ by the assumption of no correlation between them. Thus,

$$\Phi_{yx}(s) = \hat{a}(s)\Phi_{xx}(s), \tag{18.4}$$

where we have defined the "cross-power" or "cross-power density," $\Phi_{yx}(s) =< \hat{y}\hat{x}^* >$. Eq. (18.4) can be solved for

$$\hat{a}(s) = \frac{\Phi_{yx}(s)}{\Phi_{xx}(s)} \tag{18.5}$$

and Fourier inverted for a_n.

Define

$$C_{yx}(s) = \frac{\Phi_{yx}(s)}{\sqrt{\Phi_{yy}(s)\Phi_{xx}(s)}}. \tag{18.6}$$

C_{yx} is called the "coherence"; it is a complex function, whose magnitude it is not hard to prove is $|C_{yx}(s)| \leq 1.$[5] Substituting into (18.5) we obtain,

$$\hat{a}(s) = C_{yx}(s) \sqrt{\frac{\Phi_{yy}(s)}{\Phi_{xx}(s)}}. \tag{18.7}$$

Thus the phase of the coherence is the phase of $\hat{a}(s)$. If the coherence vanishes in some frequency band, then so does $\hat{a}(s)$ and in that band of frequencies, there is no relationship between y_n, x_n. Should $C_{yx}(s) = 1$, then $y_n = x_n$ in that band of frequencies. (Beware that some authors use the term "coherence" for $|C_{yx}|^2$.)

Noting that

$$\Phi_{yy}(s) = |\hat{a}(s)|^2 \Phi_{xx}(s) + \Phi_{nn}(s), \tag{18.8}$$

and substituting for $\hat{a}(s)$,

$$\Phi_{yy}(s) = \left| C_{yx}(s) \sqrt{\frac{\Phi_{yy}(s)}{\Phi_{xx}(s)}} \right|^2 \Phi_{xx}(s) + \Phi_{nn}(s) = |C_{yx}(s)|^2 \Phi_{yy}(s) + \Phi_{nn}(s) \tag{18.9}$$

Or,

$$\Phi_{yy}(s) \left(1 - |C_{yx}(s)|^2 \right) = \Phi_{nn}(s) \tag{18.10}$$

That is, the fraction of the power in y_n at frequency s, *not related to* x_m, is just $\left(1 - |C_{yx}(s)|^2 \right)$, and is called the "incoherent" power. It obviously vanishes if $|C_{yx}| = 1$, meaning that in that band of frequencies, y_n would be perfectly calculable (predictable) from x_m. Alternatively,

$$\Phi_{yy}(s) |C_{yx}(s)|^2 = |\hat{a}(s)|^2 \Phi_{xx}(s) \tag{18.11}$$

which is the fraction of the power in y_n that is related to x_n. This is called the "coherent" power, so that the total power in y is the sum of the coherent and incoherent components. These should be compared to the corresponding results for ordinary correlation above.

Estimation

As with the power densities, the coherence has to be estimated from the data. The procedure is essentially the same as estimating e.g., $\Phi_{xx}(s)$. One has observations x_n, y_n. The $\tilde{\Phi}_{xx}(s), \tilde{\Phi}_{yy}(s)$ are estimated from the products $\hat{x}(s_n) \hat{x}(s_n)^*$, etc. as above. The cross-power is estimated from products $\hat{y}(s_n) \hat{x}(s_n)^*$; these are then averaged for statistical stability in the frequency domain (frequency band-averaging) or by prior multiplication of each by one of the multitapers, etc., and the sample coherence obtained from the ratio

$$\tilde{C}_{yx}^{\nu}(s) = \frac{\tilde{\Phi}_{yx}^{\nu}(s)}{\sqrt{\tilde{\Phi}_{yy}^{\nu}(s) \tilde{\Phi}_{xx}^{\nu}(s)}}. \tag{18.12}$$

Exercise. Let $y_n = x_n + (1/2) x_{n-1} + \theta_n$, where θ_n is unit variance white noise. Find, analytically, the coherence, the coherent and incoherent power as a function of frequency and plot them.

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[5]Consider $< (\hat{x} + \lambda \hat{y})(\hat{x} + \lambda \hat{y})^* >= \langle |\hat{x}|^2 \rangle + \lambda \langle \hat{y}\hat{x}^* \rangle + \lambda^* \langle \hat{y}^* \hat{x} \rangle + |\lambda|^2 \langle |\hat{y}|^2 \rangle \geq 0$ for any choice of λ. Choose $\lambda = -\langle \hat{x}^* \hat{y} \rangle^2 / \langle |\hat{y}|^2 \rangle$ and substitute. One has $1 - \langle \hat{x}\hat{y}^* \rangle^2 / \langle |\hat{x}|^2 \rangle \langle |y|^2 \rangle \geq 0$.
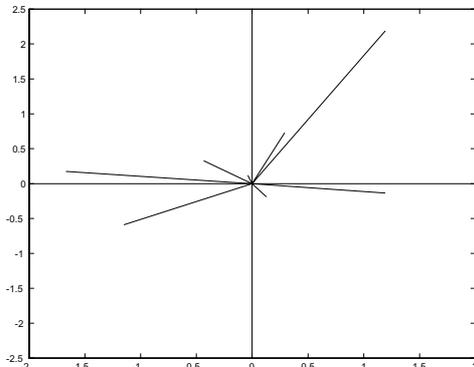
FIGURE 23. The coherence calculation involves summing vectors to produce a dominant direction (determined by the coherence phase) and amplitude determined by the degree of coherence. Here the true coherence $\gamma = 0$, and $\nu = 8$ random vectors are being summed. That they would sum to a zero-length vector is very improbable. As $\nu \to \infty$, the expected length would approach zero.

The Blackman-Tukey method would estimate $\tilde{\Phi}_{yx}^{\nu}(s)$ via the Fourier transform of the truncated (windowed) sample cross-covariance:

$$\tilde{\Phi}_{yx}^{\nu}(s) = \mathcal{F}\left(w_{\tau} \frac{1}{N} \sum_{n=0}^{N-1} x_n y_{n+\tau}\right),$$

in a complete analogy with the computation by this method of $\tilde{\Phi}_{xx}^{\nu}(s)$, etc. Again, the method should be regarded as primarily of historical importance, rather than something to be used routinely today.

$\tilde{C}_{yx}^{\nu}(s)$ is a somewhat complicated ratio of complex random variates and it is a problem to estimate its probability density. As it is a complex quantity, and as the magnitude, $\left|\tilde{C}(s)\right|$, and phase, $\tilde{\phi}(s)$, have different physical characteristics, the probability densities are sought for both. The probability density for the amplitude of the sample coherence was studied and tabulated by Amos and Koopmans (1962). As this reference is not so easily obtained, a summary of the results are stated here. One has to consider two time series for which the true coherence magnitude, at frequency s is denoted γ. Then note first, the estimate (18.12) is biassed. For example, if $\gamma = 0$ (no coherence), the expected value $\left|\left\langle \tilde{C}_{yx}^{\nu}(s)\right\rangle\right| > 0$. That is, the sample coherence for two truly incoherent time series is expected to be greater than zero with a value dependent upon ν. As $\nu \to \infty$, the bias goes to zero. More generally, if the coherence is finite, $\left|< \tilde{C}_{yx}^{\nu}(s) >\right| > \gamma$, there is a tendency for the sample coherence to be too large.

The reasons for the bias are easy to understand. Let us suppose that the cross-power and auto-power densities are being estimated by a local frequency band averaging method. Then consider the calculation of $\tilde{\Phi}_{yx}^{\nu}(s)$, as depicted in Fig. 23 under the assumption that $\gamma = 0$. One is averaging a group of vectors in the complex plane of varying magnitude and direction. Because the true coherence is zero, the theoretical average should be a zero magnitude vector. But for any finite number ν of such vectors, the probability
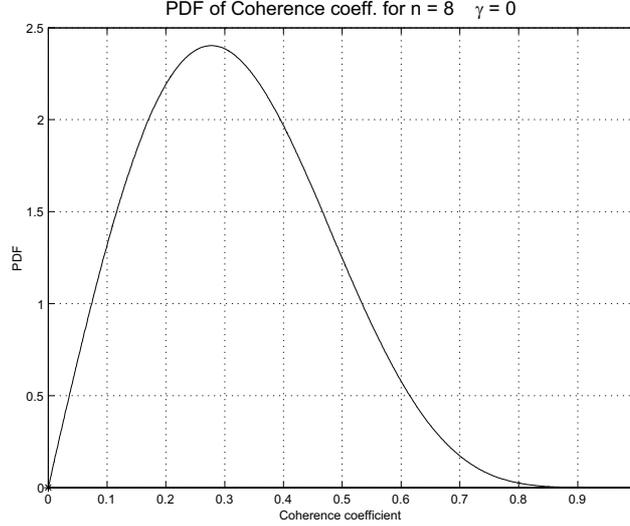
FIGURE 24. Probability density for the sample coherence magnitude, $|C_{yx}(s)|$, when the true value, $\gamma = 0$ for $\nu = 8$. The probability of actually obtaining a magnitude of zero is very small, and the expected mean value is evidently near 0.3.

that they will sum exactly to zero is vanishingly small. One expects the average vector to have a finite length, producing a sample coherence $\tilde{C}_{yx}^{\nu}(s)$ whose magnitude is finite and thus biassed. The division by $\sqrt{\tilde{\Phi}_{yy}^{\nu}(s)\,\tilde{\Phi}_{xx}^{\nu}(s)}$ is simply a normalization to produce a maximum possible vector length of 1.

In practice of course, one does not know γ, but must use the estimated value as the best available estimate. This difficulty has led to some standard procedures to reduce the possibility of inferential error. First, for any ν, one usually calculates the bias level, which is done by using the probability density

$$p_C(Z) = \frac{2\left(1-\gamma^2\right)^{\nu}}{\Gamma(\nu)\,\Gamma(\nu-1)} Z\left(1-Z^2\right)^{\nu-2} \sum_{k=0}^{\infty} \frac{\gamma^{2k}\Gamma^2(\nu+k)\,Z^{2k}}{\Gamma^2(k+1)} \tag{18.13}$$

for the sample coherence amplitude for $\gamma = 0$ and which is shown in Fig. 24 for $\nu = 8$. A conventional procedure is to determine the value C_0 below which $\left|\tilde{C}_{yx}^{\nu}(s)\right|$ will be confined 95% of the time. C_0 is the "level of no significance" at 95% confidence. What this means is that for two time series, which are completely uncorrelated, the sample coherence will lie below this value about 95% of the time, and one would expect, on average for about 5% of the values to lie (spuriously) above this line. See Fig. 25. Clearly this knowledge is vital to do anything with a sample coherence.

If a coherence value lies clearly above the level of no significance, one then wishes to place an error bound on it. This problem was examined by Groves and Hannan (1968; see also Hannan, 1970). As expected, the size of the confidence interval shrinks as $\gamma \to 1$ (see Fig. 26).

The probability density for the sample coherence phase depends upon γ as well. If $\gamma = 0$, the phase of the sample coherence is meaningless, and the residual vector in Fig. 23 can point in any direction at
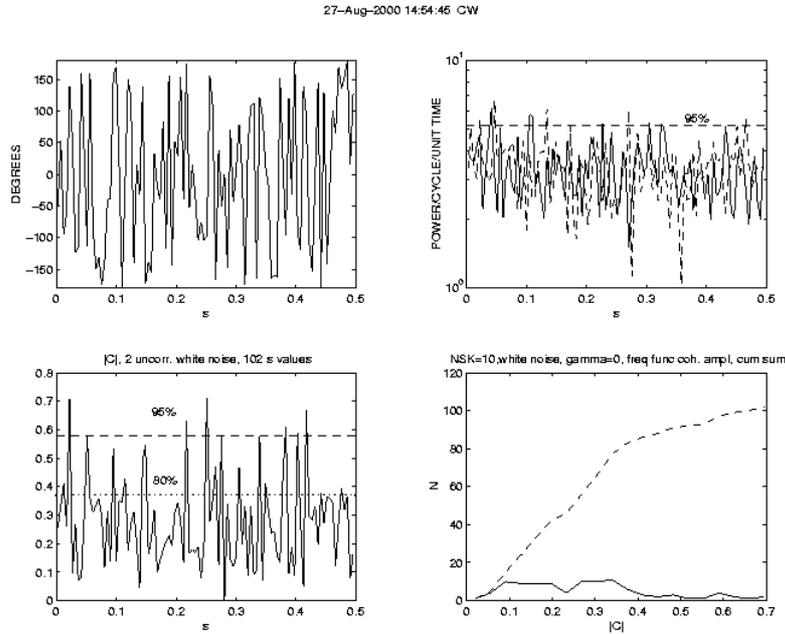
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FIGURE 25. Power densities (upper right) of two incoherent (independent) white noise processes. The theoretical value is a constant, and the horizontal dashed line shows the value below which 95% of all values actually occur. The estimated coherence amplitude and phase are shown in the left two panels. Empirical 95% and 80% levels are shown for the amplitude (these are quite close to the levels which would be estimated from the probability density for sample coherence with $\gamma = 0$). Because the two time series are known to be incoherent, it is apparent that the 5% of the values above the 95% level of no significance are mere statistical fluctuations. The 80% level is so low that one might be tempted, unhappily, to conclude that there are bands of significant coherence—a completely false conclusion. For an example of a published paper relying on 80% levels of no significance, see Chapman and Shackleton (2000). Lower right panel shows the histogram of estimated coherence amplitude and its cumulative distribution. Again the true value is 0 at all frequencies. The phases in the upper left panel are indistinguishable from purely random, uniformly distributed, $-\pi \leq \tilde{\phi} \leq \pi$.

all—a random variable of range $\pm\pi$. If $\gamma = 1$, then all of the sample vectors point in identical directions, the calculated phase is then exactly found, and the uncertainty vanishes. In between these two limits, the uncertainty of the phase depends upon γ, diminishing as $\gamma \to 1$. Hannan (1970) gives an expression
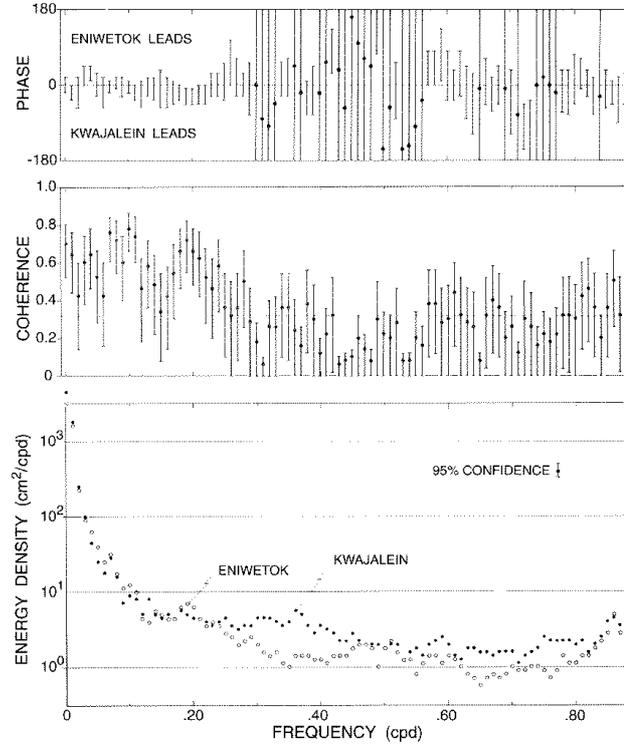
FIGURE 26. (from Groves and Hannan, 1968). Lower panel displays the power density spectral estimates from tide gauges at Kwajalein and Eniwetok Islands in the tropical Pacific Ocean. Note linear frequency scale. Upper two panels show the coherence amplitude and phase relationship between the two records. A 95% level-of-no-significance is shown for amplitude, and 95% confidence intervals are displayed for both amplitude and phase. Note in particular that the phase confidence limits are small where the coherence magnitude is large. Also note that the confidence interval for the amplitude can rise above the level-of-no-significance even when the estimated value is itself below the level.

for the confidence limits for phase in the form (his p. 257, Eq. 2.11):

$$\left| \sin \left[\tilde{\phi}\left(s\right) - \phi\left(s\right) \right] \right| \leq \left[\frac{1 - \tilde{\gamma}^2}{\left(2\nu - 2\right)\tilde{\gamma}^2} \right] t_{2\nu-2}\left(\alpha\right). \tag{18.14}$$

Here $t_{2\nu-2}\left(\alpha\right)$ is the $\alpha\%$ point of Student's $t-$distribution with $2\nu - 2$ degrees of freedom. An alternative approximate possibility is described by Jenkins and Watts (1968, p. 380-381).

Exercise. Generate two white noise processes so that $\gamma = 0.5$ at all frequencies. Calculate the coherence amplitude and phase, and plot their histograms. Compare these results to the expected probability densities for the amplitude and phase of the coherence when $\gamma = 0$. (You may wish to first read the section below on Simulation.)
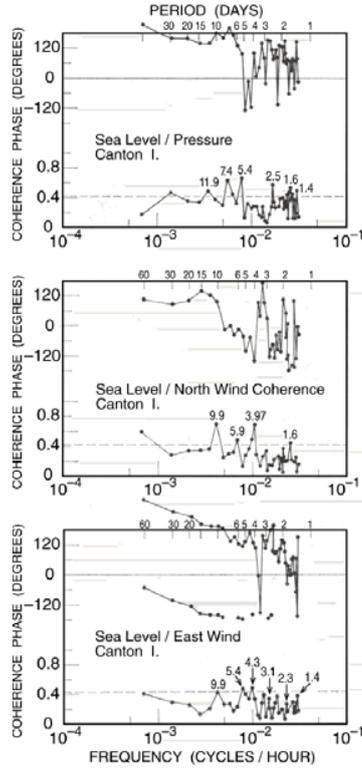
FIGURE 27. Coherence between sealevel fluctuations and atmospheric pressure, north wind and eastwind at Canton Island. An approximate 95% level of no significance for coherence magnitude is indicated. At 95% confidence, one would expect approximately 5% of the values to lie above the line, purely through statistical fluctuation. The high coherence at 4 days in the north component of the wind is employed by Wunsch and Gill (1976, from whom this figure is taken) in a discussion of the physics of the 4 day spectral peak.

Figure 27 shows the coherence between the sealevel record whose power density spectrum was depicted in Fig. 15 (left) and atmospheric pressure and wind components. Wunsch and Gill (1976) use the resulting coherence amplitudes and phases to support the theory of equatorially trapped waves driven by atmospheric winds.

The assumption of a convolution relationship between time series is unnecessary when employing coherence. One can use it as a general measure of phase stability between two time series. Consider for example, a two-dimensional wavefield in spatial coordinates $\mathbf{r} = (r_x, r_y)$ and represented by

$$\eta\left(\mathbf{r}, t\right) = \sum_{n}\sum_{m} a_{nm}\cos\left(\mathbf{k}_{nm}\cdot\mathbf{r} - \boldsymbol{\sigma}_{nm}t - \phi_{nm}\right) \qquad (18.15)$$

where the a_{nm} are random variables, uncorrelated with each other. Define $y(t) = \eta(\mathbf{r}_1, t)$, $x(t) = \eta(\mathbf{r}_2, t)$. Then it is readily confirmed that the coherence between x, y is a function of $\Delta\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, as well as σ and the number of wavenumbers $\mathbf{k}_{nm}$ present at each frequnecy. The coherence is 1 when $\Delta\mathbf{r} = 0$, and with falling magnitude with growth in $|\Delta\mathbf{r}|$. The way in which the coherence declines with growing separation can be used to deduce the number and values of the wavenumbers present. Such estimated values are part of the basis for the deduction of the Garrett and Munk (1972) internal wave spectrum.