Fall 2018 14.01 Problem Set 2 - *Solutions*

Problem 1: True or False (24 points)

For each of the following statements, indicate if they are True or False. Justify your answer.

1. (4 points) Suppose potatoes are a Giffen good. When the price of potatoes increases, both the substitution and the income effects cause the demand for potatoes to increase.
   Solution: False, the substitution effect always leads to a decrease in demand, but in the case of a Giffen good the income effect is sufficiently strong to lead to an increase in demand.

2. (4 points) John consumes only sodas (x) and pizza (y). Suppose that his preferences can be represented by $U = \min\{x, y\}$, so sodas and pizza are perfect complements: John wants to drink one soda for every pizza he eats. If the price of sodas increases, John's demand for both sodas and pizzas will decrease, and all the change in demand will be due to the substitution effect.
   Solution: False, all the change in demand is due to income effect.

3. (4 points) Consider a situation with only two goods, x and y. It is possible for one of the goods to be inferior, but they can't both be inferior at the same time.
   Solution: True, it is not possible for both goods to be inferior at the same time. If that were the case, when income increases the consumer would spend a lower total amount of money than before, which contradicts non-satiation. Note: extra credit to be given if non-satiation is explicitly mentioned as a necessary assumption for this to hold.

4. (4 points) A production function uses capital and labor as inputs. If both capital and labor have diminishing marginal returns, then the production function can't display increasing returns to scale.
   Solution: False, for example $F(K, L) = K^{0.6}L^{0.6}$ has diminishing marginal returns and increasing returns to scale.

5. (4 points) Returns to scale for a firm are the same at all production levels.
   Solution: False, returns to scale may change as production increases, for example from increasing to decreasing.

6. (4 points) Thomas Malthus' prediction about mass starvation was wrong because the diminishing marginal product of labor has been offset by the increase in agricultural productivity.
   Solution: True, Malthus ignored productivity improvements, acreage of land is fixed but arability is not.

Problem 2 (35 points)

Anne consumes only books (x) and video games (y). Her preferences can be represented by the following utility function: $U = x^2 y$. The price of books is p_x, the price of video games is p_y, and Anne has an income of m dollars.

1. (5 points) Write down Anne's budget constraint, calculate the Marginal Rate of Substitution (at an arbitrary bundle (x, y)) and compute her demand for books and video games(as a function of p_x, p_y and m).

   Solution: The budget constraint is

$$p_x x + p_y y \leq m \tag{1}$$

   Also correct if written as an equality.

   The MRS is

$$\begin{aligned} MRS &= -\frac{2xy}{x^2} \qquad (2) \\ &= -2\frac{y}{x} \qquad (3) \end{aligned}$$

   Setting MRS equal to the MRT we get

$$\begin{aligned} 2\frac{y}{x} &= \frac{p_x}{p_y} \qquad (4) \\ y &= \frac{p_x x}{2p_y} \qquad (5) \end{aligned}$$

   Replacing in the budget constraint

$$p_x x + p_y \frac{p_x x}{2p_y} = m \tag{6}$$

$$\Rightarrow x\left(p_x, p_y, m\right) = \frac{2}{3}\frac{m}{p_x} \tag{7}$$

$$y\left(p_x, p_y, m\right) = \frac{1}{3}\frac{m}{p_y} \tag{8}$$

2. (5 points) Compute the price elasticity of the demand for books? Compute the cross-price elasticity of the demand for books with respect to the price of video games. (The cross-price elasticity measures the change of the quantity demanded for a good in response to a change in the price of another good, holding everything else constant. It is measured as the percentage change in quantity demanded for the first good divided by the percentage change in price of the second good.)

   Solution: The price elasticity of books is

   $$\frac{\partial x}{\partial p_x}\frac{p_x}{x} = -1$$

   and the cross-price elasticity is

   $$\frac{\partial x}{\partial p_y}\frac{p_y}{x} = 0$$

3. (5 points) Draw the Engel curve for video games. Are video games an inferior or a normal good?
   Solution: Video games are a normal good.

4. (5 points) Suppose that initially the prices are $p_x = p_y = 1$ and income is $m = 90$. How many books does Anne buy? Now suppose that the price of books increases to $p_x = 2$, how many books will she buy now? How much of the drop in demand for books is due to the substitution effect and how much is due to the income effect? Calculate this numerically and show it in a graph.

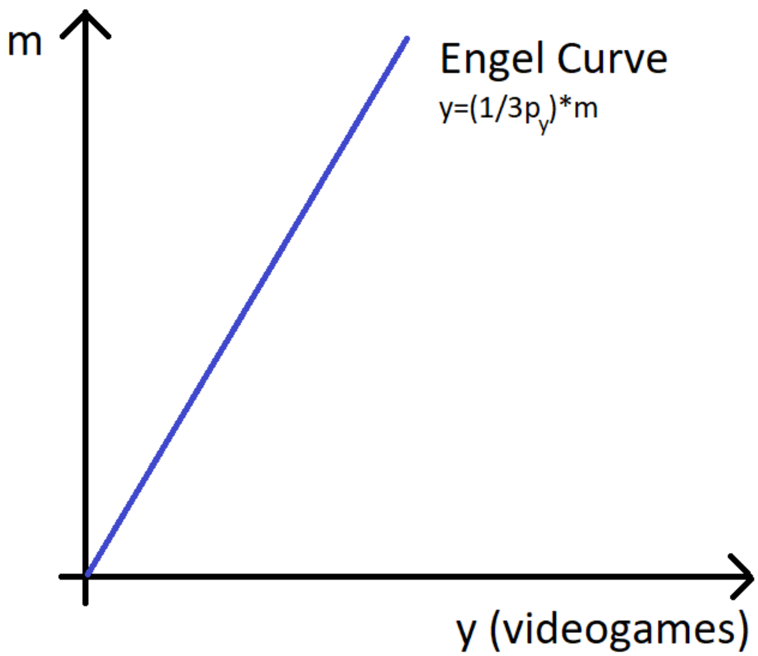
   Solution: At the initial prices, the demands are

   $$x = \frac{2}{3}90 = 60 \tag{9}$$

   $$y = \frac{1}{3}90 = 30 \tag{10}$$

   and at the final prices the demands are

Figure 1:



$$x = \frac{2}{3}\frac{90}{2} = 30 \quad (11)$$
$$y = \frac{1}{3}90 = 30 \quad (12)$$

To compute the substitution and income effects, we need to find the bundle $(\tilde{x}, \tilde{y})$ *such that the consumer gets the same utility as with the initial bundle but the MRS is equal to the new price ratio:*

$$\begin{cases} \tilde{x}^2\tilde{y} = 60^2 30 \\ 2\frac{\tilde{y}}{\tilde{x}} = 2 \end{cases}$$

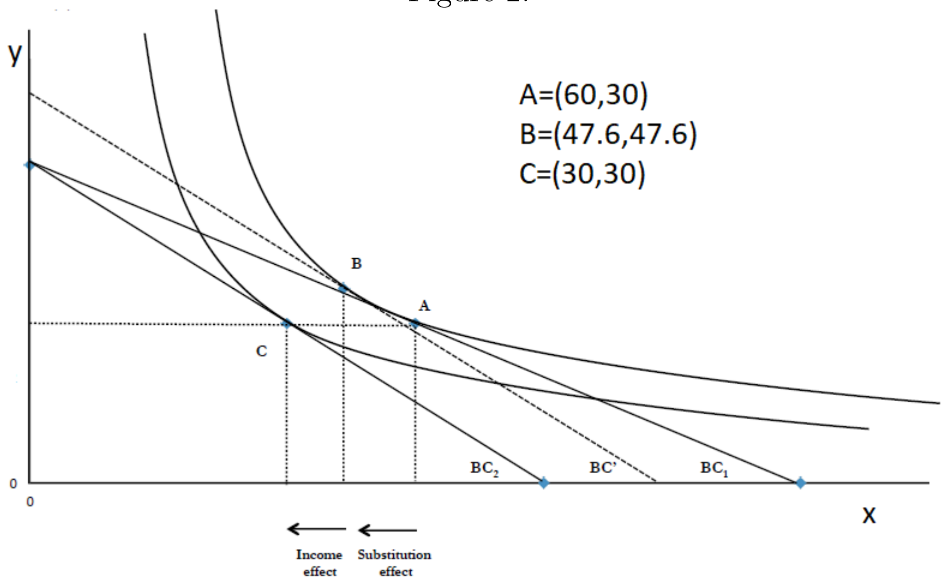
so we have that

$$\tilde{x} = \tilde{y} = 30 \times 2^{\frac{2}{3}} \cong 47.6$$

Therefore, the substitution effect accounts for a drop in demand for books of $(60 - 47.6) = 12.4$ *units, while the rest, 17.6 units, is due to income effect.*

5. (15 points) Suppose that her utility function were $u(x, y) = x^{1/2} + y$ instead. Prices are $p_x = 1$, $p_y = 10$ and her income is $m = 50$. (i) Calculate her demand for

Figure 2:



books and video games. (*ii*) Now suppose that the price of video games increases to $p_y = 15$ how many books and video games will she buy now? What is different in this case compared to question 2.4. Explain. (*iii*) How much of the drop in demand for video games is due to the substitution effect and how much is due to the income effect? Calculate this numerically and show it in a graph.

Solution: (i) The budget constraint is

$$x + 10y \leq 50 \tag{13}$$

The MRS is

$$MRS = -\frac{x^{-1/2}}{2} \tag{14}$$

$$\tag{15}$$

Setting MRS equal to the MRT we get

$$-\frac{x^{-1/2}}{2} = -\frac{p_x}{p_y} \tag{16}$$

$$x\left(p_x, p_y, m\right) = \frac{p_y^2}{4p_x^2} = \frac{10^2}{4 \cdot 1} = 25 \tag{17}$$

Substituting the demand for x back in the budget constraint yields the demand for y

$$y=\frac{m}{p_y}-\frac{p_x}{p_y}\frac{p_y^2}{4p_x^2}\rightarrow y=\frac{m}{p_y}-\frac{p_y}{4p_x}=5/2 \tag{18}$$

(ii)

In this case, solving the same way will yield a negative quantity for y which means that we have a corner solution. Hence, we set

$$y=0\rightarrow x=m/p_x=50 \tag{19}$$

Intuitively she would like to consume even more x (books) compared to y (video games) but she can't consume any less y.

To compute the substitution and income effects, we need to find the bundle $(\tilde{x},\tilde{y})$ such that the consumer gets the same utility as with the initial bundle but the MRS is equal to the new price ratio:

$$\begin{cases} \tilde{x}^{1/2}+\tilde{y}=25^{1/2}+2.5 \\ -\frac{1}{2}\tilde{x}^{-1/2}=-\frac{1}{15} \end{cases}$$

so we have that

$$\tilde{x}=\frac{225}{4} \tag{20}$$

$$\tilde{y}=25^{1/2}+\frac{5}{2}-\frac{15}{2}=0 \tag{21}$$

Therefore, the substitution effect crowds out consumption of y entirely. No change in the consumption of video games is due to the income effect. Note: Also correct if the income and substitution effects for books were computed.

Problem 3 (18 points)

For each of the following production functions:

$$\begin{aligned} (a)\ \ F(L,K) &= L^2K^{\frac{1}{2}} \\ (b)\ \ F(L,K) &= L+L^{\frac{1}{2}}K^{\frac{1}{2}} \\ (c)\ \ F(L,K) &= 2L+K \end{aligned}$$

1. (2 points, per production function) Find the marginal product of labor and capital, and state if the returns to capital and labor are increasing, decreasing or constant.

2. (2 points, per production function) Find the marginal rate of technical substitution.

3. (2 points, per production function) State whether the production function exhibits constant, increasing or decreasing returns to scale.

Solution:
Function (a) :

$$\begin{aligned} MP_L &= 2LK^{\frac{1}{2}} \\ MP_K &= \frac{1}{2}L^2K^{-\frac{1}{2}} \end{aligned}$$

The returns to labor are increasing while the returns to capital are decreasing.

$$MRTS = -\frac{MP_L}{MP_K} = -\frac{4K}{L}$$

To analyze returns to scale, we multiply all inputs by a constant $s > 1$:

$$F(sL, sK) = s^{\frac{5}{2}}F(L, K) > sF(L, K)$$

so there are IRS.
Function (b) :

$$\begin{aligned} MP_L &= 1 + \frac{1}{2}L^{-\frac{1}{2}}K^{\frac{1}{2}} \\ MP_K &= \frac{1}{2}L^{\frac{1}{2}}K^{-\frac{1}{2}} \end{aligned}$$

so there are decreasing marginal returns to both labor and capital.

$$MRTS = -\frac{MP_L}{MP_K} = -\frac{1 + \frac{1}{2}L^{-\frac{1}{2}}K^{\frac{1}{2}}}{\frac{1}{2}L^{\frac{1}{2}}K^{-\frac{1}{2}}}$$

To analyze returns to scale, we multiply all inputs by a constant $s > 1$:

$$\begin{aligned} F(sL, sK) &= sL + sL^{\frac{1}{2}}K^{\frac{1}{2}} \\ &= sF(L, K) \end{aligned}$$

so there are CRS.
Function $(c) : 2L + K$

$$\begin{aligned} MP_L &= 2 \\ MP_K &= 1 \end{aligned}$$

so there are constant returns to both capital and labor.

$$MRTS = -\frac{MP_L}{MP_K} = -2$$

To analyze returns to scale, we multiply all inputs by a constant $s > 1$:

$$\begin{aligned} F(sL, sK) &= 2sL + sK \\ &= sF(L, K) \end{aligned}$$

so there are CRS.

Problem 4 (24 points)

Firm X has the following production function $f(K, L) = (L + K^{2/3})$.

1. (6 points) Find the marginal product of labor and capital and the marginal rate of technical substitution, and state if the returns to capital and labor are increasing, decreasing or constant. Explain what is "weird" about this MRTS.

   Solution:

   $$\begin{aligned} MP_L &= 1 \\ MP_K &= \frac{2}{3}K^{-\frac{1}{3}} \end{aligned}$$

   The returns to labor are constant while the returns to capital are decreasing.

   $$MRTS = -\frac{MP_L}{MP_K} = -\frac{3}{2}K^{1/3}$$

   The MRTS does not depend on L because the production function is quasi-linear in L.

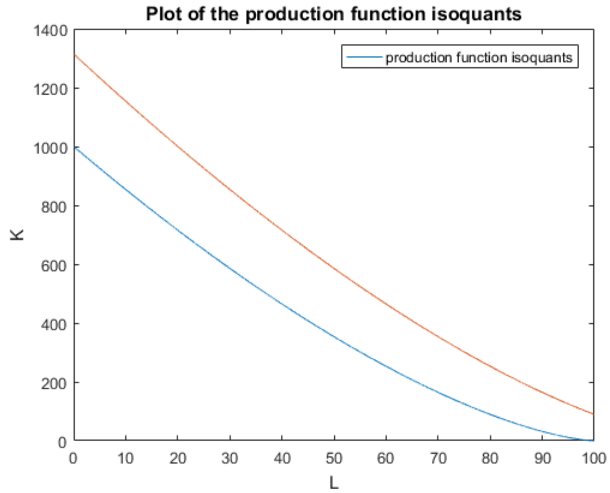
2. (8 points) Graph the isoquants of the production function and state whether the production function exhibits constant, increasing or decreasing returns to scale.

   Solution:

   The production function is quasi-linear in L. The isoquants are decreasing and convex with the slope at each point not depending on L.

   $$q = f(K, L) = L + K^{2/3} \tag{22}$$

Figure 3:



To analyze returns to scale, we multiply all inputs by a constant $s > 1$:

$$F(sL, sK) = sL + s^{2/3}K$$
$$< s(L + K^{2/3}) = sF(L, K)$$

so there are DRS.

3. (9 points) Now suppose that the production function is $f(K, L) = exp(2t)(L + K^{2/3})$ where t measures years. *(a)* (4 points) How does this affect the marginal products of labor and capital, the returns to scale and the MRTS? *(b)* (5 points) What is the rate of productivity increase over time? Graph the isoquants for $q = 100$ for $t = 1, 2$. Explain.

   Solution: (a)

$$\begin{aligned} MP_L &= exp(2t) \\ MP_K &= exp(2t)\frac{2}{3}K^{-\frac{1}{3}} \end{aligned}$$

The marginal product of labor and capital are now increasing over time.

$$MRTS = -\frac{MP_L}{MP_K} = -\frac{3}{2}K^{1/3}$$

The MRTS on the other hand is not affected by the increase in productivity.

To analyze returns to scale, we multiply all inputs by a constant $s > 1$:

$$F(sL, sK) = exp(2t)(sL + s^{2/3}K)$$
$$< exp(2t)s(L + K^{2/3}) = sF(L, K)$$

so there are still DRS i.e. the returns to scale are not affected by the productivity improvement.

(b)

The rate of productivity improvement is defined as:

$$r(t) = \frac{dA/dt}{A} = 2 \tag{23}$$

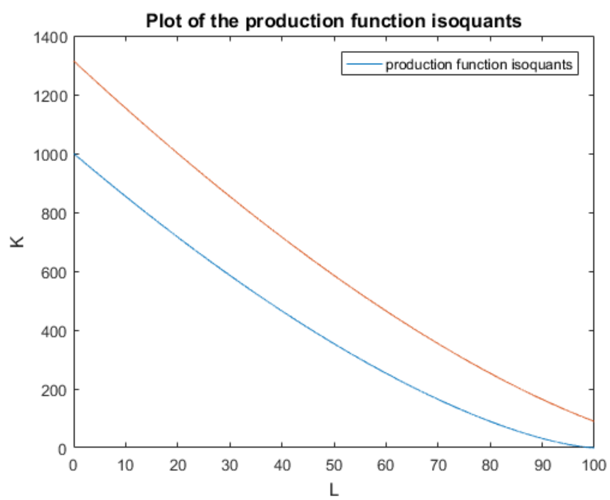
so output doubles each year. It is also correct if written in discrete time i.e.

$$r_t = \frac{A_{t+1} - A_t}{A_t} \tag{24}$$
$$= \frac{exp(2(t+1)) - exp(2t)}{exp(2t)} = e^2 - 1 \tag{25}$$

The reason the growth rates differ significantly in continuous and discrete time is because the time interval is large, $\Delta t = 1$ *and so the approximation is not good.*

The isoquants move inwards now. The same quantity can be produced with lower quantities of K *and* L *due to the increase in productivity from* $t = 1$ *to* $t = 2$.

Figure 4:



Plot of the production function isoquants
production function isoquants
1400
1200
1000
800
600
400
200
0
K
0
10
20
30
40
50
60
70
80
90
100
L