Problem Set 7 Solutions:
Econ 14.04

1. (a) Let q be the probability of the column player picking $Left$:

    1. Suppose that the row player plays bottom. Any mixing strategy of Left and Center will dominate this
    2. Suppose that the row player plays middle. Any strategy that puts weight $q > 1/3$ dominates Right.
    3. Suppose that the row player plays top. Any strategy that puts weight $q < 2/3$ dominates Right.

    Thus any strategy with $q \in [1/3, 2/3]$ dominates

   (b) By the same logic as 1, playing bottom is dominated for the Row player once Right is eliminated as a strategy. We now look at reaction functions.

$$\begin{aligned} r_{Row}(Left) &= Middle \\ r_{Row}(Center) &= Top \\ r_{Col}(Top) &= Left \\ r_{Col}(Center) &= Center \end{aligned}$$

	$Left$	$Center$
Top	$-1, \underline{3}$	$\underline{3}, -1$
$Middle$	$\underline{3}, -1$	$-1, \underline{3}$

   Since the best responses cycle, there is no nash equilibrium in dominant strategies (ie, my strategy doesn't change with changes in the strategy of the other player.

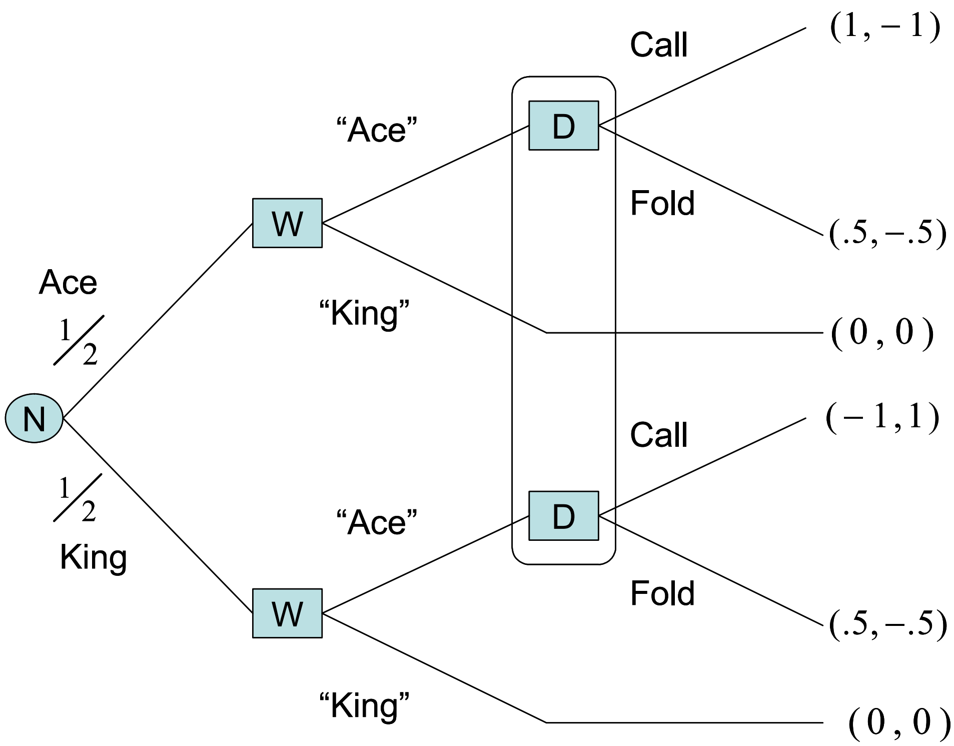
   (c) Let q be the probability of choosing left. Let p be the probability of choosing up. For a mixed strategy to be in equilibrium, the expected value of each strategy must be the same. Thus:

$$\begin{aligned} [(-1)q + 3(1-q)] &= [3q + (-1)(1-q)] \\ [(-1)p + 3(1-p) &= [3p + (-1)(1-p)] \end{aligned}$$

   Reducing yields:

$$\begin{aligned} q &= \frac{1}{2} \\ p &= \frac{1}{2} \end{aligned}$$

2. Starting with the extensive form:



The extensive form starts with nature randomly picking Ace or King. Will's strategy depends on nature, thus his strategy will have two parts: Given that nature picks Ace what is his strategy? Given that nature picks King, what is his strategy? Thus, Will's strategy will be a pair ("Ace","Ace") the first denoting his strategy when nature picks Ace, the second when Nature picks King.

Since Davy lacks information, he does not know what state he is in when making his selection (denoted by the big circle around his decision. His opponent has announced Ace but he does not know if he actually holds an Ace or a King. His strategy is thus confined to the case where Will calls "Ace" and is just a single action - Call or Fold.

We can now move on to the normal form. Will has four possible strategies: ("Ace","Ace"), ("Ace","King"), ("King","Ace"),("King","King"). Davy has two Strategies: (Call),(Fold). We calculate the expected value of outcome given Will's pair of actions and Davy's single action:

	$Call$	$Fold$
$("Ace","Ace")$	$0, 0$	$.5, -.5$
$("Ace","King")$	$.5, -.5$	$.25, -.25$
$("King","Ace")$	$-.5, .5$	$.25, -.25$
$("King","King")$	$0, 0$	$0, 0$

Notice that $("Ace","King")$ dominates $("King","King")$ and $("Ace","Ace")$ dominates $("Ace","King")$. The game can now be represented by a simple 2x2 game:

	$Call$	$Fold$
$("Ace","Ace")$	$0, \underline{0}$	$\underline{.5}, -.5$
$("Ace","King")$	$\underline{.5}, -.5$	$.25, \underline{-.25}$

1. b. From the 2x2 game, we see that the best responses cycle just like problem 1.

   c. We can find the mixed strategies just like (1). Let p be the probability of $("Ace","Ace")$ and q the probability of $Call$:

$$
\begin{aligned}
[0q + .5(1-q)] &= [.5q + (.25)(1-q)] \\
[0p - .5(1-p) &= [-.5p - .25(1-p)]
\end{aligned}
$$

Reducing these yields:

$$.5 - .5q = .25q + .25 \rightarrow q = \frac{1}{3}, p = \frac{1}{3}$$

d. This should of read "Given that Will announces an Ace, how often should Davy call him. We solved this in the last problem: $p = \frac{1}{3}$.

3. (a) The reaction functions of the three agents are:

$$\begin{aligned} r_1(q_2, q_3) &= \frac{60 - q_2 - q_3}{2} \\ r_2(q_1, q_3) &= \frac{60 - q_1 - q_3}{2} \\ r_1(q_1, q_2) &= \frac{60 - q_1 - q_2}{2} \end{aligned}$$

(b) Since the game is symmetric, $q_1 = q_2 = q_3$, thus $q_1 = q_2 = q_3 = 15$.

(c) If firm 2 and 3 merge, the reaction functionns are:

$$\begin{aligned} r_1(q_{2+3}) &= \frac{60 - q_{2+3}}{2} \\ r_{2+3}(q_1) &= \frac{60 - q_1}{2} \end{aligned}$$

Again by symmetry, $q_{2+3} = q_1$ and thus $q_{2+3} = q_1 = 20$. The first firm is worse off since $20^2 = 400$ and $2 * 15^2 = 450$. If all 3 firms merged - $q = p = 30$. Profits would be 900 which is larger than the premerger profits of 675.

(d) If firm 1 can commit to an amount in advance, it plugs in the response functions of 2+3 and then solves its profits:

$$\max_{q_1}(60 - q_1 - r_2(q_1) - r_3(q_1))q_1$$

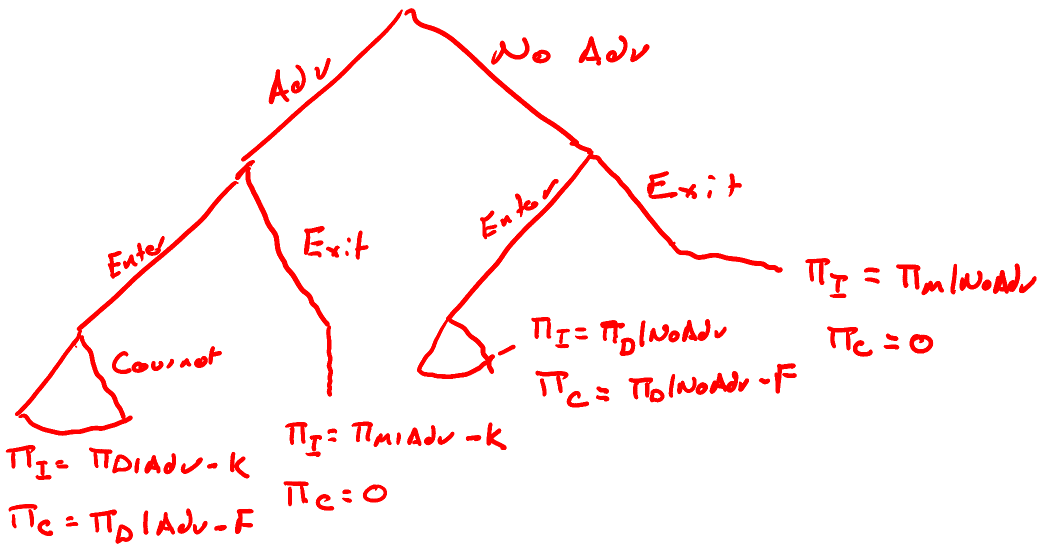
Given a level of output q_1, the best response functions of r_2 and r_3 is to produce $\frac{60-q_1}{3}$ units. Thus, the first firm maximizes:

$$\left(60 - q_1 - \frac{60 - q_1}{3} - \frac{60 - q_1}{3}\right) q_1$$

FOC:

$$\begin{aligned} 20 - \frac{2}{3}q_1 &= 0 \\ q_1 &= 30 \end{aligned}$$

In this case, $q_2 = q_3 = 10$. firm 1 makes $\pi = 300$ while the other two firms make 100. This is a case of first mover advantage. It comes about because the firms are strategic compliments - when I produce more, you produce less. Thus I get an advantage by choosing my amount first and forcing you to reduce in response.

4. (a) 

(b) The Cournot duopoly equilibrium solution for advertising is:

$$q_{Adv,Cournot} = \frac{60}{3} = 20$$

$$P_{adv}(20,20) = 20$$

$$\begin{aligned} \pi_I|Cournot, Advertise &= 400 - K \\ \pi_C|Cournot, Advertise &= 400 - F \end{aligned}$$

The Cournot Duopoly Eqm for not advertising is:

$$\begin{aligned} q_{noAdv,Cournot} &= 18 \\ P_{noadv}(18,18) &= 18 \end{aligned}$$

$$\begin{aligned} \pi_I|Cournot, NoAdv &= 324 \\ \pi_C|Cournot, Advertise &= 324 - F \end{aligned}$$

The monopoly equilibrium for advertising is:

$$\begin{aligned} q_{adv,Monopoly} &= p_{adv,Mon} = 30, \pi_{mon,adv} = 900 - K \\ q_{noadv,Mon} &= p_{noadv,Mon} = 24, \pi_{mon,noadv} = 24^2 = 576 \end{aligned}$$

(c) With F = 350, it is not profitable for the second firm to enter the market when firm one doesn't advertise. Thus for any cost F, it is profitable for firm one not to advertise and be a monopoly in period 2.

(d) When F = 100, the other firm will enter the market regardless of advertising. The incumbent will advertise if K $\leq 400 - 324 = 76$.

5. (a) $\max_{x_i} k - x_i + \frac{a}{N}\sum_{i=1}^{n} x_i$
FOC: $\frac{\alpha}{N} = 1 \rightarrow \alpha = N$ is the cutoff. When $\alpha < n$, no one invests anything.

(b) $\max_{x_i} \sum_{j=1}^{n}(k - x_j + \frac{a}{N}\sum_{i=1}^{n} x_i)$
FOC: Cutoff: $\alpha = 1$. Thus when $\alpha < 1$ no one should invest, otherwise should invest.

(c) One solution is to designate the first x_i of benefit to the person who contributed it and distribute the rest evenly. This will work in part d as well.

(d) A different solution that works is to pay an agent αx_i. Note that this solution keeps all the inequality intact - it may not be optimal for someone who cares about welfare redistribution.