14.12 Game Theory – Midterm II
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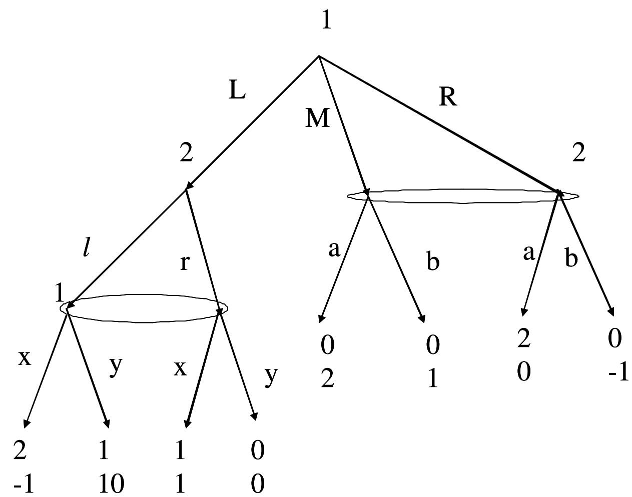
Instructions. This is an open book exam; you can use any written material. You have one hour and 20 minutes. Each question is 25 points. Good luck!

1. Compute all the subgame-perfect equilibria in pure strategies for the following game:



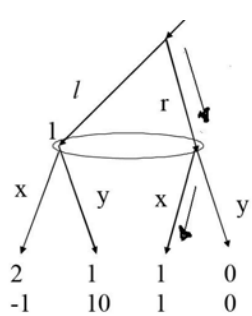
ANSWER:

First, notice that the game has two subgames: a proper subgame originated when player 1 plays L, and the whole game itself.

Since we are looking for SPEs, we need to make sure that the equilibria we find are NE of every subgame. So then, lets restrict our search to NE of the proper subgame first. This game can be represented by

	l	r
x	(2,-1)	(1,1)
y	(1,10)	(0,0)

where the underlined values are best responses. Then, we can see that the only NE of this subgame is (x,r):



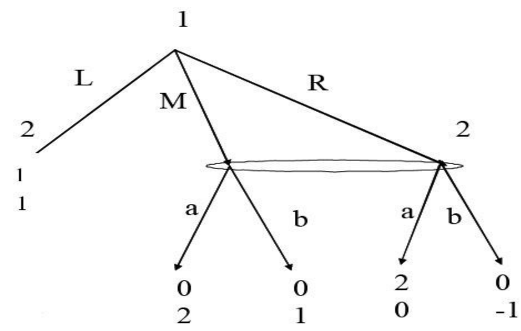


Figure 1:

Having solved for the NE of this subgame, the game reduces to the second figure above.

This game is represented by

	a	b
L	(1,$\underline{1}$)	($\underline{1}$,$\underline{1}$)
M	(0,$\underline{2}$)	(0,1)
R	($\underline{2}$,$\underline{0}$)	(0,-1)

where again the BR are underlined. We can see that this game has 2 NE, which are the SPE we are looking for. They can be written as (Lx,br), and (Rx, ar).

2 Consider the infinitely repeated game with the following stage game:

	Chicken	Lion
Chicken	3,3	1,4
Lion	4,1	0,0

All the previous actions are observed, and each player maximizes the discounted sum of his stage payoff with discount factor $\delta = 0.99$. For each strategy profile below check if it is a subgame-perfect equilibrium. (You need to state your arguments clearly; you will not get any points for Yes or No answers.)

(a) (10 points) There are two modes: Cooperation and Fight. The game starts in the Cooperation mode. In Cooperation mode, each player plays Chicken. If both players play Chicken, then they remain in the Cooperation mode; otherwise they go to the Fight mode in the next period. In the Fight mode, both play Lion, and they go back to the Cooperation mode in the following period (regardless of the actions).

ANSWER:

It is not SPE because in the fight mode each player has an incentive to deviate. In the fight mode, according to the strategy profile both players play Lion and get 0 and then go back to the cooperation mode where they both get 3 forever. This yields present value of $0 + 3\delta/(1-\delta)$ to each player. If a player deviates in the fight mode and plays chicken, his payoff is 1 in period t and then they will still go back to the cooperation in period t+1 and obtain 3 forever. This yields a higher present value of $1 + 3/(1-\delta)$.

b) (15 points) There are three modes: Cooperation, P1 and P2. The game starts in the Cooperation mode. In the Cooperation mode, each player plays Chicken. If they play (Chicken, Chicken) or (Lion, Lion), then they remain in the Cooperation mode in the next period. If player i plays Lion while the other player plays Chicken, then in the next period they go to Pi mode. In Pi mode player i plays Chicken while the other player plays Lion; they then go back to Cooperation mode (regardless of the actions).

ANSWER: We use the single-deviation principle to check if this is a SPE. In the cooperation mode according to the strategy profile both players will always play Chicken. This yields present value of $3/(1-\delta)$ to each player. If player i deviates in period t and plays Lion, his payoff increases to 4 in that period. In period $t+1$, according to the strategy profile they go to Pi mode. In Pi mode player i plays Chicken while the other player plays Lion. Then in the next period they go back to the Cooperation mode. This yields a present value of $4 + 1\delta + 3\delta^2/(1-\delta)$. Players don't want to deviate if $3/(1-\delta) > 4 + 1\delta + 3\delta^2/(1-\delta) \Leftrightarrow \delta > 1/2$.Since P1 and P2 modes are symmetrical we only have to verify for one of these modes. In mode P1 player 1 plays Chicken while the other player plays Lion. This is a NE of the stage game. Since in the next period they go back to the Cooperation mode regardless of the actions, neither player wants to deviate. If player 1 does not deviate his present payoff is $1 + 3\delta/(1-\delta)$. If he deviates and plays Lion his present payoff is $0 + 3\delta/(1-\delta)$. Hence, player 1 does not want to deviate. We can easily see that player 2 also does not want to deviate in the P1 mode. If he plays Lion his present value payoff is $4 + 3\delta/(1-\delta)$, while if he deviates and plays chicken his present value payoff is $3 + 3\delta/(1-\delta)$.

3) Consider the infinitely repeated game with the following stage game (Linear Bertrand duopoly). Simultaneously, Firms 1 and 2 choose prices $p_1 \in [0,1]$ and $p_2 \in [0,1]$, respectively. Firm i sells

$$q_i(p_1, p_2) = \begin{cases} 1 - p_i & \text{if } p_i < p_j \\ (1-p_i)/2 & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

units at price p_i, obtaining the stage payoff of $p_i q_i(p_1, p_2)$. (All the previous prices are observed, and each player maximizes the discounted sum of his stage payoffs with discount factor $\delta \in (0,1)$.) For each strategy profile below, find the range of parameters under which the strategy profile is a subgame-perfect equilibrium.

a) (10 points) They both charge $p_i = 1/2$ until somebody deviates; they both charge 0 thereafter. (You need to find the range of δ.)

ANSWER: If nobody has deviated before:

Payoff to not deviate: 1/8, 1/8, ...$\Rightarrow$ $1/8(1-\delta)$

Payoff to deviate: Notice that the only profitable deviation occurs by undercutting the price and the most profitable undercutting is just to charge infinitesmall less than your competition. You will get something very close to 1/4 by doing this so 1/4,0,0,0,...$\Rightarrow$ 1/4

So we need $1/8(1-\delta) > 1/4 \Rightarrow \frac{1}{2} > 1-\delta \Rightarrow \delta > \frac{1}{2}$

We don't get any meaningful restrictions from the histories with previous deviations.

b) (15 points) There are $n+1$ modes: Collusion, the first day of war (W_1), the second day of war (W_2), ..., and the nth day of war (W_n). The game starts in the Collusion mode. They both charge $p_i = 1/2$ in the Collusion mode and $p_i = p^*$ in the war modes (W_1,..., W_n), where $p^* < 1/2$. If both players charge what they are supposed to charge, then the Collusion mode leads to the Collusion mode, W_1 leads to W_2, W_2 leads to W_3, ..., W_{n-1} leads to W_n, and W_n leads to the Collusion mode. If any firm deviates from what it is supposed to charge at any mode, then they go to W_1. (Every deviation takes us to the first day of a new war.) (You need to find inequalities with δ, p^*, and n.)

ANSWER: If nobody has ever deviated before:

Payoff to not to deviate: 1/8 ,1/8, 1/8,...$\Rightarrow 1/8(1-\delta)$

Payoff to deviate: 1/4, p*(1-p*)/2, p*(1-p*)/2,...,p*(1-p*)/2,1/8, 1/8,...

$\Rightarrow \frac{1}{4} + \frac{p^*(1-p^*)\delta(1-\delta^n)}{2(1-\delta)} + \frac{\delta^{n+1}}{8(1-\delta)}$

so one condition we have: $\frac{1}{8(1-\delta)} > \frac{1}{4} + \frac{p^*(1-p^*)\delta(1-\delta^n)}{2(1-\delta)} + \frac{\delta^{n+1}}{8(1-\delta)}$

If we are in a war mode: Notice that we don't have to check for all the war modes. Because the lowest cost of deviattion happens in the first war mode (W_1) and the benefit of deviation in a war mode is always the same.

Payoff to not to deviate: p*(1-p*)/2, p*(1-p*)/2,...,p*(1-p*)/2,1/8, 1/8,...

$\Rightarrow \frac{p^*(1-p^*)(1-\delta^n)}{2(1-\delta)} + \frac{\delta^n}{8(1-\delta)}$

Payoff to deviate: The most profitable deviation is once again to undercut your opponent by an infinitesmall amount. This will result in a payoff that is approximately:$p^*(1-p^*)$

p*(1-p*),p*(1-p*)/2, p*(1-p*)/2,...,p*(1-p*)/2,1/8, 1/8,...

$\Rightarrow p^*(1-p^*) + \frac{p^*(1-p^*)\delta(1-\delta^n)}{2(1-\delta)} + \frac{\delta^{n+1}}{8(1-\delta)}$

So our second condition is:

$\frac{p^*(1-p^*)(1-\delta^n)}{2(1-\delta)} + \frac{\delta^n}{8(1-\delta)} > p^*(1-p^*) + \frac{p^*(1-p^*)\delta(1-\delta^n)}{2(1-\delta)} + \frac{\delta^{n+1}}{8(1-\delta)}$

4 The players in the following game are Alice, who is an MIT senior looking for a job, and Google. She has also received a wage offer r from Yahoo, but we do not consider Yahoo as a player. Alice and Google are negotiating. They use alternating offer bargaining, Alice offering at even dates $t = 0, 2, 4, \ldots$ and Google offering at odd dates $t = 1, 3, \ldots$. When Alice makes an offer w, Google either accepts the offer, by hiring Alice at wage w and ending the bargaining, or rejects the offer and the negotiation continues. When Google makes an offer w, Alice

- either accepts the offer w and starts working for Google for wage w, ending the game,
- or rejects the offer w and takes Yahoo's offer r, working for Yahoo for wage r and ending the game,
- or rejects the offer w and then the negotiation continues.

If the game continues to date $\bar{t} \leq \infty$, then the game ends with zero payoffs for both players. If Alice takes Yahoo's offer at $t < \bar{t}$, then the payoff of Alice is $r\delta^t$ and the payoff of Google is 0, where $\delta \in (0,1)$. If Alice starts working for Google at $t < \bar{t}$ for wage w, then Alice's payoff is $w\delta^t$ and Google's payoff is $(\pi - w)\delta^t$, where

$$\pi/2 < r < \pi.$$

(Note that she cannot work for both Yahoo and Google.)

(a) (10 points) Compute the subgame perfect equilibrium for $\bar{t} = 4$. (There are four rounds of bargaining.)

ANSWER:

- (2.5pts) Consider $t = 3$. Alice will get w if she accepts Google, r if she accepts Yahoo, and 0 if she rejects and continues. Thus, she must choose

$$s_{A,3} = \begin{cases} Google & \text{if } w \geq r \\ Yahoo & \text{otherwise.} \end{cases}$$

Given this, Google gets 0 if $w < r$ and $\pi - w$ if $w \geq r$. Therefore, it must choose

$$w_3 = r.$$

- (2.5pts) Consider $t = 2$. Google will get $\pi - w$ if it accepts an offer w by Alice and $\pi - w_3$ next day if it rejects the offer. Hence Google must

$$\text{Accept iff } (\pi - w) \geq \delta(\pi - w_3) \text{ i.e. } w \leq \pi(1-\delta) + \delta r.$$

The best reply for Alice is to offer

$$w_2 = \pi(1-\delta) + \delta r.$$

- (2.5pts) [**This is the most important step. Disturbingly, the majority of the students failed at this step.**] Consider $t = 1$. Consider Alice's decision. Alice will get w if she accepts Google, r if she accepts Yahoo, and δw_2 if she rejects and continues. We nned to check whether she prefers Yahoo's offer to continuing. Note that

$$r > \delta w_2 = \pi\delta(1-\delta) + \delta^2 r \iff r > \frac{\pi\delta(1-\delta)}{1-\delta^2} = \frac{\pi\delta}{1+\delta}.$$

Since $r > \pi/2 > \frac{\pi\delta}{1+\delta}$, this implies that $r > \delta w_2$. That is, Alice prefers Yahoo's offer to continuing, and hence she will never reject and continue. Therefore, she must choose

$$s_{A,1} = s_{A,3} = \begin{cases} Google & \text{if } w \geq r \\ Yahoo & \text{otherwise.} \end{cases}$$

Google then must offer $w_1 = r$.

- (2.5 pts) Consider $t = 0$. It must be obvious now that it is the same as $t = 2$. Google Accepts iff $w \leq w_2$ and Alice offers

$$w_0 = w_2 = \pi(1-\delta) + \delta r.$$

(b) (15 points) Take $\bar{t} = \infty$. Conjecture a subgame-perfect equilibrium and check that the conjectured strategy profile is indeed a subgame-perfect equilibrium.

ANSWER:

From part (a), it is easy to conjecture that the following is a SPE:

s^* : At an odd date Alice accepts an offer w iff $w \geq r$, otherwise she takes Yahoo's offer. Google offers $w_G = r$. At an even date Alice offers $w_A = \pi(1-\delta) + \delta$, and Google accepts an offer w iff $w \leq w_A$.

We use single-deviation principle to check that s^* is indeed a SPE. There are 4 major cases two check:

- Consider the case Alice is offered w.
  - Suppose that $w \geq w_G \equiv r$. Alice is supposed to accept and receive w today. If she deviates by rejecting w and taking Yahoo's offer, she will get r, which is not better that w. If she deviates by rejecting and continuing, she will offer w_A at the next day, which will be accepted. The present value of this is $\delta w_A = \pi\delta(1-\delta) + \delta^2 r < r \leq w$, i.e. this deviation yields even a lower payoff.
  - Suppose that $w < w_G \equiv r$. Alice is supposed to reject it and take Yahoo's offer with payoff r. If she deviates accepting w, she will get the lower payoff of $w < r$. If she deviates by rejecting and continuing, she will get w_A next day, with a lower present value of $\delta w_A = \pi\delta(1-\delta) + \delta^2 r < r$.
- Consider a case Google offers w. If $w \geq r$, it will be accepted, yielding a payoff of $\pi - w$ to Google. If $w < r$, then Alice will go to Yahoo, with payoff of 0 to Google. Therefore, the best response is to offer $w = r > 0$, as in s^*. There is no profitable (single) deviation.
- Consider the case Google is offered w.
  - Suppose that $w \leq w_A$. If Google deviates and rejects, it will pay δ tomorrow with payoff $\delta(\pi - r) = (\pi - w_A)$, which is not better than $\pi - w_A$.
  - Suppose that $w > w_A$. If Google deviates and accepts, then it will get only $\pi - w$, while it would get the present value of $\delta(\pi - r) = (\pi - w_A)$ by rejecting the offer.
- Consider a node in which Alice offers. Google will accept iff $w \leq w_A$. If she offers $w > w_A$ she gets r next day, with present value of $\delta r < w_A$. Therefore, the best reply is to offer $w = w_A$, and there is no profitable deviation.

[In part (b) most important cases are the acceptance/rejection cases, especially that of Alice. Many of you skipped those cases, and wrongly concluded that a non-SPE profile is a SPE.]