14.12 Game Theory – Midterm II
11/15/2011

Prof. Muhamet Yildiz

Instructions. This is a **closed book** exam. You have 90 minutes. You need to show your work when it is needed. All questions have equal weights. You may be able to receive partial credit for stating the *relevant* facts, such as the definition of the solution concept, towards the correct solution. Also, if you leave the answer for a part blank or just write "I don't know the answer", you will receive 10% of the full grade for that part. Good luck!

1. This question assesses your knowledge of single deviation principle.

   (a) State the single-deviation principle. (Be as precise as you can, but you do not need to state the assumptions on the game.)

   Solution: In a multistage game that is continuous at infinity, a strategy profile is a subgame-perfect Nash equilibrium if and only if it passes the single-deviation test at every stage for every player. Single-deviation test is: Consider a strategy profile s^*. Pick any stage (after any history of moves). Assume that we are at that stage. Pick also a player i who moves at that stage. Fix all the other players' moves as prescribed by the strategy profile s^* at the current stage as well as in the following game. Fix also the moves of player i at all the future dates, but let his moves at the current stage vary. Can we find a move at the current stage that gives a higher payoff than s^*, given all the moves that we have fixed? If the answer is Yes, then s^* fails the single-deviation test at that stage for player i.

   (b) State the single-deviation principle in the context of an infinitely repeated game.

   Solution: In an infinitely repeated game, one uses the *single-deviation principle* in order to check whether a strategy profile is a subgame-perfect Nash equilibrium. In such a game, single-deviation principle takes a simple form and is applied through *augmented stage games.* Here, augmented refers to the fact that one simply augments the payoffs in the stage game by adding the present value of future payoffs under the purported equilibrium. One may also use the term *reduced game* instead of augmented stage game, interchangeably.

   (c) Find a subgame-perfect Nash equilibrium of the following infinite-horizon bargaining game and verify that it is indeed a subgame-perfect Nash equilibrium. At each t, one of the players is randomly selected as the proposer; Player 1 is selected with probability 1/3, Player 2 is selected with probability 1/3 and with the remaining probability no player is selected. If no player is selected, nothing happens that day and they proceed to $t+1$. If a player is selected as a proposer, he proposes a division $(x, 1-x) \in [0,1]^2$ and the other player accepts or rejects. If the proposal is accepted, the game ends with payoffs $\left(\delta^t x, \delta^t (1-x)\right)$. Otherwise, they proceed to $t+1$. Each player gets 0 if no proposal is ever accepted.

   Hint: Consider an equilibrium in which each i always proposes $(x_i, 1-x_i)$, which is accepted, for some $x_1, x_2 \in [0,1]$.

   Solution: Let V be the value of the stage game before we see who is selected as a proposer (or none of them). Similarly, let V_p be the value of the stage game for the proposer, V_n be the value of the stage game if the other player is the proposer. Then, we have $V = \frac{1}{3}V_p + \frac{1}{3}V_n + \frac{1}{3}\delta V$. Since we divide one at the stage game, $V_p + V_n = 1$. Solving, we get $V = \frac{1}{3-\delta}$. Following the hint, we consider a strategy that the proposer offers $1 - x_0$ to the other player and keeps x_0 for himself. As the value of rejecting and continuing to the next round is δV, we have $1 - x_0 = \delta V$. Note that in this setting, we have $x_0 = V_p$, $1 - x_0 = V_n$. Thus, we need to show that the following strategy profile is SPE: for both players, if a player is chosen as a proposer he proposes $\frac{3-2\delta}{3-\delta}$ for himself and $\frac{\delta}{3-\delta}$ for the other player. If the other player is a proposer, accepts the offer iff offered at least $\frac{\delta}{3-\delta}$.

Let's verify that this is indeed SPE. For player 1 (P1), if he is the proposer, his offer will be rejected if he offers less than $\frac{\delta}{3-\delta}$. In this case, game moves to the next round and his expected payoff is $\delta V = \frac{\delta}{3-\delta} \leq \frac{3-2\delta}{3-\delta}$, so he is better off by offering $\frac{\delta}{3-\delta}$ and keeps $\frac{3-2\delta}{3-\delta}$ for himself. Clearly, there is no point in offering more than $\frac{\delta}{3-\delta}$. If P2 is the proposer, if he rejects his expected payoff from going to the next round is $\delta V = \frac{\delta}{3-\delta}$, so he has no incentive to deviate from the strategy. By symmetry, the same argument works for P2 and and this completes the proof.

2. Alice is an art dealer, and Bob and Carroll are two art collectors. There is a painting that is worth v_A, v_B, and v_C for Alice, Bob, and Carroll, respectively, where $0 < v_A < v_B < v_C$. First, Alice sets a reserve price $r \geq 0$. Then, observing r, Bob and Carroll simultaneously submit bids $b_B \in [0, v_B]$ and $b_C \in [0, v_C]$, respectively. If the highest bid is less than r, then Alice keeps the painting, and each player gets 0. If the highest bid is at least r, then the highest bidder wins the auction; *if* $b_B = b_C \geq r$, *then Carroll wins the auction*. The winner pays his own bid to Alice and gets the painting. (Writing i for the winner and j for the other art collector, the payoffs are $b_i - v_A$ for Alice, $v_i - b_i$ for i, and 0 for j.)

   (a) Find all the subgame-perfect Nash equilibria in pure strategies.

   Solution: For any $r \leq v_B$, the only Nash equilibrium in the auction is (v_B, v_B). For any $r > v_B$, in any Nash equilibrium, $b_C = r$, and of course $b_B \leq v_B$. Any such pair is a Nash equilibrium. Then, Alice chooses $r = v_C$. In sum, Alice sets $r = v_C$ and Bob chooses any $b_B \in [0, v_B]$ if $r > v_B, b_B = v_B$ if $r \leq v_B$. For Carroll, he bids any $b_C \in [0, v_C]$ if $r > v_C$, $b_C = v_B$ if $r \leq v_B$, and r if $v_B < r \leq v_C$. Note that you should write a complete strategy profile for your answer.

   (b) How would your answer change if a bidder is allowed to bid above his own value?

   Solution: When $b_B > v_B$ is allowed, any pair (b, b) with $b \geq \max\{r, v_B\}$ is a Nash equilibrium in the subgame. Any $(\hat{r}, b(\cdot), b(\cdot))$ with $b(r) \geq \max\{r, v_B\}$ for all r and $b(\hat{r}) = v_C$ is a SPE. All lead to same outcome as the one in part (a).

3. Consider the infinitely repeated game in which the stage game is the game in Problem 2 and the discount factor is $\delta > 0$. Fix some $\hat{r} \in (v_A, v_B)$. Assuming that δ is sufficiently high, find a subgame-perfect Nash equilibrium with the following outcome path: $r = \hat{r} = b_C > b_B$ at even dates $t \in \{0, 2, \ldots\}$ and $r = \hat{r} = b_B > b_C$ at odd dates $t \in \{1, 3, \ldots\}$. Specify the range of δ under which your strategy profile is a subgame-perfect Nash equilibrium, and verify that this is indeed the case.

Solution: Consider the following strategy profile. For any t and any h at the beginning of t in which no bidder deviated from the following rule, they follow the following rule:

$$\begin{aligned} r(h) &= \hat{r} \\ b_C(h,r) &= \min\{r,\hat{r}\}, b_B(h,r) < \min\{r,\hat{r}\} \qquad \text{(if } t \text{ is even)} \\ b_B(h,r) &= \min\{r,\hat{r}\}, b_C(h,r) < \min\{r,\hat{r}\} \qquad \text{(if } t \text{ is odd)}; \end{aligned}$$

for any other h, the players play according to SPE in part (a) of the previous problem. (That is, if the bidders deviate from the path, they switch to the stage game SPE; they do not switch if Alice deviates.) After the switch, they play a SPE of the stage game forever; hence the stratgy profile is a SPE after the switch. Before the switch, given the bidders' strategy, Alice's move does not affect her future payoff and she can sell the painting at price $\hat{r}$ at most. Hence, in the augmented stage game, setting the reservation price $r = \hat{r}$ is a best response to the bidders' strategies. Now consider any h before the switch and any reserve price r. In the augmented stage game, the payoff of a bidder i is as follows, assuming that the other bidder follows the equilibrium strategy:

case	payoff from strategy	best deviation payoff
$b_i(h,r) = \min\{r,\hat{r}\}, r \le \hat{r}$	$(v_i - r)(1-\delta) + \delta^2 V_i$	$(v_i - r)(1-\delta)$
$b_i(h,r) = \min\{r,\hat{r}\}, r > \hat{r}$	$\delta^2 V_i$	$(v_i - r)(1-\delta)$
otherwise	δV_i	$(v_i - \max\{r,\hat{r}\})(1-\delta)$

where $V_i = (v_i - \hat{r})/(1+\delta)$. In order for the strategy profile to be a SPE, i must not have an incentive to deviate in all cases above. A sufficient condition is

$$\delta^2 (v_i - \hat{r})/(1+\delta) \ge (v_i - \hat{r})(1-\delta),$$

i.e.,

$$\delta^2 \ge 1 - \delta^2,$$

i.e., $\delta \ge 1/\sqrt{2}$.

4. Consider a two-player game with the following payoff matrix

	L	R
a	$\theta, \theta+\gamma$	$0, 0$
b	$0, \gamma$	θ, θ

where $\theta \in \{1,3\}$ is privately known by Player 1 and $\gamma \in \{-2,2\}$ is privately known by Player 2. Moreover,

$$\begin{aligned} \Pr(\theta = 3, \gamma = 2) &= \Pr(\theta = 1, \gamma = -2) = 1/3 \\ \Pr(\theta = 3, \gamma = -2) &= \Pr(\theta = 1, \gamma = 2) = 1/6. \end{aligned}$$

(a) Write this formally as a Bayesian game.
$N = \{1,2\}$
$T_1 = \{3,1\}, T_2 = \{2,-2\}$
$A_1 = \{a,b\}, A_2 = \{L,R\}$
$p(t_1 = 3, t_2 = 2) = p(t_1 = 1, t_2 = -2) = 1/3$
$p(t_1 = 3, t_2 = -2) = p(t_1 = 1, t_2 = 2) = 1/6$
And utility functions given by the payoff matrix.

(b) Compute a Bayesian Nash equilibrium. (Verify that it is indeed a Bayesian Nash equilibrium.)

There are 2 BNE. bb, RR is a Bayesian Nash Equilibrium.

Player 2's type -2 has expected value $E[\theta] > 0$ so he will not deviate. Player 2's type 2 has expected value $E[\theta] = 7/3$, so he does not deviate either. Player 1 will not deviate because $\theta > 0$.

The second BNE is type $\theta = 3$ plays a, $\theta = 1$ plays b, $\gamma = 2$ plays L and $\gamma = -2$ plays R. To see that this is BNE, check deviations.

Player 1's type $\theta = 3$ plays 1 and believes that player 2 has type $\gamma = 2$ and plays L with probability 2/3, and $\gamma = -2$ and plays R with probability 1/3. His payoff from a is $3 \cdot 2/3 + 0 \cdot 1/3 = 2$ and from b is $0 \cdot 2/3 + 3 \cdot 1/3 = 1$.

He does not deviate. $\theta = 1$ has payoff 1/3 from a and 2/3 from b, so he does not deviate.

Player 2's type 2 has payoff 4 for playing L and 1/3 from playing R. Player 2's type $\gamma = -2$ has payoff -1 from playing L and 2/3 from playing R so he does not deviate.