Chapter 3

Attitudes Towards Risk

The previous lectures explored the implications of expected utility maximization. In this lecture, considering the lotteries over money, I will introduce the basic notions regarding risk, such as risk aversion and certainty equivalence. These concepts play central role in most areas of modern economics.

3.1 Theory

Take the set of alternatives as $X = \mathbb{R}$ which corresponds to the wealth level of the decision maker. The decision maker has an increasing von Neumann-Morgenstern utility function $u : \mathbb{R} \to \mathbb{R}$, representing his preferences over the lotteries on his wealth level. I will assume that u is differentiable whenever needed. Since we have a continuum of consequences, it is more convenient to represent lotteries by cumulative distribution functions $F : X \to [0, 1]$. I write f for the density of F when it exists. The expected utility of F is given by

$$U(F) \equiv E_F(u) \equiv \int u(x)\, dF(x),$$

where E_F is the expectation operator under F. The expected wealth level under F is

$$E_F(x) = \int x dF(x).$$

By comparing $E_F(x)$ to $E_F(u)$, one can learn about the decision maker's attitudes towards risk.

A decision maker is called *risk averse* if he always prefers sure wealth level $E_F(x)$ to the lottery F, i.e.,

$$E_F(u) \leq u(E_F(x)) \qquad (\forall F).$$

He is called *strictly risk averse* if the inequality is always strict for nondegenerate lotteries. He is called *risk neutral* if he is always indifferent:

$$E_F(u) = u(E_F(x)) \qquad (\forall F).$$

Finally, he is called *risk seeking* (or risk loving) if he prefers lottery to the sure outcome, i.e.,

$$E_F(u) \geq u(E_F(x)) \qquad (\forall F).$$

Clearly, by Jensen's inequality, which you must know by now, risk aversion corresponds to the concavity of the utility function:

- DM is risk averse if and only if u is concave;
- he is strictly risk averse if and only if u is strictly concave;
- he is risk neutral if and only if u is linear, and
- he is risk seeking if and only if u is convex.

Another way to assess the attitudes towards risk is certainty equivalence. The *certainty equivalent* of a lottery F, denoted by $CE(F)$, is a sure wealth level that yields the same expected utility as F. That is,

$$CE(F) = u^{-1}(U(F)) = u^{-1}(E_F(u)).$$

It is immediate from the definitions that

- DM is risk averse if and only if $CE(F) \leq E_F(x)$ for all F;
- he is risk neutral if and only if $CE(F) = E_F(x)$ for all F, and
- he is risk seeking if and only if $CE(F) \geq E_F(x)$ for all F.

It is sometimes useful to quantify the degree of risk aversion. There are two important measures of risk aversion. The first one is *absolute risk aversion*:

$$r_A(x) = -u''(x)/u'(x),$$

which is also called Arrow-Pratt coefficient of absolute risk aversion. Note that u'' measures the concavity of the utility function, while u' normalizes the concavity as the utility representation is unique up to affine transformations.

A convenient assumption in economic analysis is *constant absolute risk aversion* (CARA). A CARA utility function takes the simple form of

$$u(x) = -e^{-\alpha x},$$

where α is the coefficient of absolute risk aversion. This utility function becomes especially convenient when the lotteries are distributed normally. In that case, the certainty equivalent becomes

$$CE(F) = \mu - \frac{1}{2}\alpha\sigma^2$$

where μ and σ^2 are the mean and the variance of the distribution, respectively. While CARA is a convenient assumption, some may find it more plausible that absolute risk aversion is decreasing with wealth level (DARA), so that richer people take higher risks.

Indeed, some may want to normalize the amount of risk aversion with respect to the level of wealth. This leads to the concept of *relative risk aversion*. The *coefficient of relative risk aversion* is

$$r_R(x) = -xu''(x)/u'(x).$$

The *constant relative risk aversion* (CRRA) utility function takes the form of

$$u(x) = x^{1-\rho}/(1-\rho),$$

where ρ is the coefficient of constant relative risk aversion. When $\rho = 1$, it is the log utility function: $u(x) = \log(x)$.

Using the above concepts, one can also compare the attitudes of two decision makers towards risk. To this end, take any two decision makers DM1 and DM2 with u_1 and u_2 and write $CE_i(F) \equiv u_i^{-1}(E_F(u_i))$ and $r_{A,i} = -u_i''/u_i'$ for the certainty equivalent and coefficient of absolute risk aversion under u_i for $i \in \{1, 2\}$.

Definition 3.1 *DM1 is more risk averse than DM2 if either of the equivalent conditions in the next proposition holds.*

Proposition 3.1 *The following are equivalent.*

1. $u_1 = g \circ u_2$ *for some concave function* g*,*
2. $CE_1(F) \leq CE_2(F)$ *for every* F*;*
3. $r_{A,1} \geq r_{A,2}$ *everywhere.*

Proof. Since both u_1 and u_2 are increasing, there exists an increasing function g such that $u_1 = g \circ u_2$. To see the equivalence between 1 and 2, note that $CE_1(F) = u_2^{-1}(g^{-1}(E_F(g(u_2))))$. By Jensen's inequality, g is concave if and only if $E_F(g(u_2)) \leq g(E_F(u_2))$ for every F. Thus, g is concave if and only if, for every F,

$$\begin{aligned} CE_1(F) &= u_2^{-1}\left(g^{-1}(E_F(g(u_2)))\right) \\ &\leq u_2^{-1}\left(g^{-1}(g(E_F(u_2)))\right) = u_2^{-1}(E_F(u_2)) \\ &= CE_2(F), \end{aligned}$$

where the inequality uses also the fact that g^{-1} is increasing.

To see the equivalence between 1 and 3, note that

$$r_{A,1} = -\frac{u_1''}{u_1'} = -\frac{g'' \cdot (u_2')^2 + g' \cdot u_2''}{g' \cdot u_2'} = -\frac{u_2''}{u_2'} - \frac{g''}{g'}u_2' = r_{A,2} - \frac{g''}{g'}u_2'.$$

Hence,

$$g'' = \frac{g'}{u_2'} \cdot (r_{A,2} - r_{A,1}).$$

Thus, $r_{A,1} \geq r_{A,2}$ everywhere if and only if $g'' \leq 0$ everywhere, which is true if and only if g is concave. ■

Since one can envision and individual with two different initial wealths as two different decision makers, the above characterization allows one to explore how one's attitude towards risk changes as his initial wealth level changes. To do this, let us write w for the initial wealth level of an individual and write lotteries as changes in his wealth. That is, given any lottery z, the final wealth of the individual is $x = w + z$. Define $u(\cdot|w)$ by

$$u(z|w) = u(z + w).$$

The coefficient of absolute risk aversion under initial wealth w is

$$r_A(z|w) = -u''(z+w)/u'(z+w) = r_A(z+w).$$

Corollary 3.1 *The decision maker becomes less risk averse against the changes in his wealth (z) when his initial wealth increases if and only if he has decreasing absolute risk aversion.*

Proof. Note that for every fixed z and $w \geq w'$,

$$r_A(z|w) \leq r_A(z|w') \iff r_A(z+w) \leq r_A(z+w').$$

Hence, by Proposition 3.1, DM is less risk averse against the additive risks under w vis a vis a lower wealth level w' for all $w \geq w'$ iff r_A is a decreasing function. ■

One can further conclude that *if the decision maker has constant absolute risk aversion, then his attitude toward the risk in changes in his wealth (z) is independent of his initial wealth.*

Similar facts can be obtained about the decision maker's attitudes toward the risk in multiplication of his wealth, using relative risk aversion instead. To do that, write y for the multiplication of his initial wealth so that his final wealth level is $x = yw$. Define $u_y(\cdot|w)$ by

$$u_y(z|w) = u(yw).$$

The coefficient of absolute risk aversion against y under initial wealth w is

$$r_{A,y}(z|w) = -u_y''(y|w)/u_y'(y|w) = -wu''(yw)/u'(yw) = r_R(yw).$$

That is absolute risk aversion against the multiplicative risk in one's wealth is simply his relative risk aversion according to his underlying utility function at the relevant values. This immediately yields the following comparative statics.

Corollary 3.2 *DM's risk aversion against the multiplication y in his wealth is decreasing in his initial wealth w_0 if he has decreasing relative risk aversion r_R; DM's risk aversion against the multiplication y in his wealth is independent of his initial wealth w_0 if he has constant relative risk aversion r_R.*

3.2 Applications

3.2.1 Insurance

Consider a decision maker who has initial wealth of w and may lose 1 unit of his wealth with probability p. He can buy an insurance, which is a divisible good. A unit insurance costs q and covers one unit of loss in case of a loss. We want to understand his demand for insurance. Let λ be the amount of insurance he buys. His expected utility is

$$U(\lambda) = u(w - q\lambda)(1-p) + u(w - q\lambda - (1-\lambda))\, p.$$

First consider the case of actuarially unfair price $q > p$, which is natural given that the insurance company needs to cover its operational costs. In that case, he buys only a partial insurance, i.e., $\lambda < 1$. Indeed,

$$U'(1) = (p(1-q) - q(1-p))\, u'(w-q) < 0,$$

i.e., U is strictly decreasing at the full insurance level, and hence optimal λ must be less than 1. Therefore, he bears some of the risks no matter how risk averse he is and how low the mark up $q - p$ is. This is because when the amount of risk gets lower and lower, u becomes approximately linear and the decision maker becomes approximately risk neutral.

Now consider the case of $q = p$, the actuarially fair price. This case is important in the literature because it corresponds to the competitive price (assuming insurance companies do not have any other costs). In that case, he buys full insurance (i.e. $\lambda = 1$). To see this, note that under actuarially fair price, his expected wealth is $E_\lambda[x] = w - q$ for each λ. Hence, for any $\lambda < 1$,

$$CE(\lambda) < E_\lambda[x] = w - q = CE(1),$$

where $CE(\lambda)$ is the certainty equivalent of wealth when he buys λ units of insurance. Thus, $\lambda = 1$ yields higher certainty equivalence than any other λ.

Finally, consider a more risk averse decision maker with certainty equivalence operator CE'. If the former decision maker buys full insurance, so will the new one. Indeed, for any $\lambda < 1$,

$$CE'(\lambda) \leq CE(\lambda) < CE(1) = CE'(1),$$

where the first inequality is the fact that the new decision maker is more risk averse, the second inequality is by the fact that full insurance was optimal for the original decision maker and the equality is by the fact that there is no risk under full insurance.

3.2.2 Optimal Portfolio Choice

Consider a decision maker with initial wealth w. There is also a risky asset that yields z for each dollar invested. Write F for the cdf of z. We want to understand how much the decision maker would invest in the risky asset. Write α for the level of investment and α^* for the optimal α. The expected utility is

$$U(\alpha) = \int u(w + \alpha(z-1))\, dF,$$

which is a concave function. The optimal investment is determined by the first-order condition

$$U'(\alpha^*) = \int u'(w + \alpha^*(z-1))(z-1)\, dF = 0.$$

First observe that he will not take any risk if the expected return $E[z]-1$ is not positive. Indeed, if $E[z]-1 \le 0$,

$$U'(0) = \int u'(w)(z-1)\, dF = u'(w)(E[z]-1) \le 0.$$

On the other hand, he will invest a positive amount as long as any positive expected return $(E[z]-1 > 0)$:

$$U'(0) = u'(w)(E[z]-1) > 0.$$

This is, again, because he is approximately risk neutral against small risks.

A main finding in this example is that more risk averse agents invest less in the risky asset. I will show this intuitive fact formally next. Consider two decision makers DM1 and DM2 with utility functions u_1 and u_2, respectively, such that DM1 is more risk averse than DM2. Hence, $u_1 = g \circ u_2$ for some concave increasing function g with $g'(w) = 1$. Denote the variables for decision maker i by subscript i, e.g., by writing α_1^* and α_2^* for the optimal investments of DM1 and DM2, respectively. Now, for any α, since $u_1'(w + \alpha(z-1)) = g'(w + \alpha(z-1))\, u_2'(w + \alpha(z-1))$,

$$u_1'(w + \alpha(z-1)) \ge u_2'(w + \alpha(z-1)) \iff z \le 1.$$

Hence,

$$\left[u_1'\left(w+\alpha\left(z-1\right)\right)-u_2'\left(w+\alpha\left(z-1\right)\right)\right]\left(z-1\right)\leq 0$$

everywhere. Thus, for every α,

$$U_1'\left(\alpha\right)-U_2'\left(\alpha\right)=\int\left[u_1'\left(w+\alpha\left(z-1\right)\right)-u_2'\left(w+\alpha\left(z-1\right)\right)\right]\left(z-1\right)dF\leq 0.$$

Therefore, $\alpha_1^*\leq\alpha_2^*$. (One way to see this is to observe that $U_1'\left(\alpha_2^*\right)\leq U_2'\left(\alpha_2^*\right)=0$. Hence, U_1 is decreasing at α_2^* and must have been maximized at a lower value.)

Together with Corollary 3.1, the above finding yields the following monotone comparative statics on the optimal investment level as a function of initial wealth:

- if the agent has decreasing absolute risk aversion, then α^* is increasing with the initial wealth level w;
- if the agent has constant absolute risk aversion, then α^* is independent of the initial wealth level w.

The optimal level of investment as a proportion of the initial wealth is related to the relative risk aversion. To see this, write $\beta=\alpha/w$, and observe that the final wealth level is

$$x=w+\beta w\left(z-1\right)=w\cdot\left(1+\beta\left(z-1\right)\right).$$

Hence, the risk is about the multiplication $1+\beta\left(z-1\right)$ of his initial wealth. From the above finding and Corollary 3.2, we can conclude following:

- If DM has decreasing relative risk aversion, then the optimal investment level β^* as a proportion of the initial wealth w is increasing in w;
- If DM has constant relative risk aversion, then the optimal investment level β^* as a proportion of the initial wealth w is independent of w; i.e. $\alpha^*=bw$ for some constant b.

3.2.3 Optimal Risk Sharing

Consider a set of agents $N=\{1,\ldots,n\}$. Each i has a concave, differentiable, and bounded utility function u_i. There is an unknown state $s\in S$. Each agent i has a risky asset $\bar{x}_i : S \to \mathbb{R}$, whose outcome depends on the state. A feasible allocation is a list $(x_1, \ldots, x_n)$ of consumption plans $x_i : S \to \mathbb{R}$ such that

$$x_1(s) + \cdots + x_n(s) \leq \bar{x}_1(s) + \cdots + \bar{x}_n(s) \tag{3.1}$$

for each s. We want to explore the Pareto-optimal allocations. To this end, write A for the set of all feasible allocations. Note that A is a convex set. Write also

$$V = \{(E(u_1(x_1)), \ldots, E(u_n(x_n))) \,|\, (x_1, \ldots, x_n) \in A\}$$

for the set of feasible utility vectors and $\bar{V} = \{v | v \leq v' \text{ for some } v' \in V\}$ for the comprehensive closure of V. Note that since each u_i is concave and A is convex, $\bar{V}$ is also a convex set.

Now consider any Pareto-optimal allocation $x^* = (x_1^*, \ldots, x_n^*)$. By definition, the utility vector $(E(u_1(x_1^*)), \ldots, E(u_n(x_n^*)))$ is on the Pareto-frontier of the set $\bar{V}$. Since $\bar{V}$ is convex, $(E(u_1(x_1^*)), \ldots, E(u_n(x_n^*)))$ is a solution to the program

$$\max_{(v_1,\ldots,v_n)\in\bar{V}} \sum_{i\in N} \lambda_i v_i = \max_{(v_1,\ldots,v_n)\in V} \sum_{i\in N} \lambda_i v_i$$

for some vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ of positive coefficients (by the Support-Function Theorem, which is a version of the Separating-Hyperplane Theorem). Equivalently, x^* is a solution to the program

$$\max_{(x_1,\ldots,x_n)\in A} E\left[\sum_{i\in N} \lambda_i u_i(x_i)\right].$$

Hence, for each $s \in S$, $x^*(s) = (x_1^*(s), \ldots, x_n^*(s))$ is a solution to the program

$$\max_{(x_1(s),\ldots,x_n(s))} \sum_{i\in N} \lambda_i u_i(x_i(s)) \text{ subject to (3.1).} \tag{3.2}$$

That is, the Pareto-optimal risk sharing allocations can be written as a maximization of weighted sum of utilities at each state where the utility weight of individuals are independent of the state. While it is possible to compensate one individual for his loss in one state by using a higher utility weight in another state, the above finding establishes that such a compensation is not Pareto optimal. The optimality requires that we determine the allocation of the consumption at each state independent of what allocation would have been in another state.

When the utility functions are differentiable, (3.2) implies that

$$\lambda_i u_i'(x_i^*(s)) = \lambda_j u_j'(x_i^*(s)) \qquad (\forall i, j, s). \tag{3.3}$$

(This is true because $x_i(s)$ can take any value in $\mathbb{R}$.) That is, the normalized marginal utilities are the same for all players, where the normalization is independent of the state. This is quite intuitive: optimality requires the consumption good is given to the player with highest marginal value.

Under constant absolute risk aversion, this leads to a simple allocation rule. To establish this fact, assume also that the agents have constant absolute risk aversion: $u_i(x) = -e^{-\alpha_i x}$. Then, (3.3) simplifies to

$$\alpha_i x_i^*(s) = \alpha_j x_j^*(s) + \ln(\lambda_i \alpha_i) - \ln(\lambda_j \alpha_j).$$

Since the consumptions add up to $\bar{X}(s) \equiv \bar{x}_1(s) + \cdots + \bar{x}_n(s)$, this further implies that

$$x_i^*(s) = \frac{1/\alpha_i}{1/\alpha_1 + \cdots + 1/\alpha_n} \bar{X}(s) + \tau_i \qquad (\forall i, s)$$

for some state-independent transfer τ_i, where the transfers add up to 0. That is, in any optimal allocation, every player i owns $\frac{1/\alpha_i}{1/\alpha_1+\cdots+1/\alpha_n}$ fraction of each asset in addition to a state-independent transfer. Clearly, such an allocation is easily implemented by trading the assets among the agents. The crucial point is that in all optimal allocations, a player owns the same fraction of the assets—in accordance with his relative risk tolerance. The distributive concerns, such as equity, are addressed through lump-sum transfers, as the expected payoffs of individuals can be varied optimally by only transferring deterministic wealth between them.

Note also that when the assets are normally distributed (i.e. $\bar{x}_i \sim N(\bar{\mu}_i, \sigma_i^2)$), we have transferable utility in terms of certainty equivalences because $CE_i(x_i) = E[x_i] - Var(x_i)\alpha_i/2$ when x_i is normal. Then, optimality reduces to maximizing $\sum_{i \in N} CE_i(x_i)$, which is equivalent to minimizing the total risk $\sum_{i,j} \frac{1}{2}\alpha_i \lambda_{i,j}^2 \sigma_j^2$, where $\lambda_{i,j}$ is the share i owns in asset $\bar{x}_j$. This also yields the above allocation rule, where $\lambda_{i,j} = \frac{1/\alpha_i}{1/\alpha_1+\cdots+1/\alpha_n}$.

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