

# Chapter 7

## Internal Conflict & Hyperbolic Discounting

We have so far assumed a unitary decision maker who tries to choose the best alternative for himself, under various theories of what is best and how able he is. The decision makers often do not have such a coherent agenda and they make their decisions under conflicting motives. When the decision maker is a group of individuals (as in the case of a committee or a government), such a conflict is clear, and such diverse preferences cannot be represented by a single representative agent, as it has been demonstrated by Arrow's Impossibility Theorem. In that case, we better model the situation as a game theoretical problem and apply strategic analysis. Even when the decision maker is a single individual, he often faces an internal conflict between conflicting motives and temptations, and his behavior may reflect these conflicts. In this chapter, I will present some of these issues focusing on hyperbolic discounting, under which the decision maker's preferences change over time, leading to a conflict between the selves at different times.

### 7.1 Time Preferences

In general, the value of getting a dollar today is not the same as the value of getting a dollar tomorrow. The decision makers tend to discount the future payoffs so that a dollar today is more valuable than a dollar tomorrow. The standard economic theory assumes that the decision makers discount the future payoffs "exponentially". In the

simplest form, it assumes that the value of getting  $x$  at time  $t$  as perceived at date  $s \leq t$  is

$$U(x, t|s) = e^{-r(t-s)}u(x)$$

where  $u$  is a known utility function and  $r > 0$  is a known constant discount rate (e.g. interest rate). (Hence the term "exponential discounting".) The standard theory does allow  $r$  to vary by time possibly stochastically and  $u$  to depend on additional parameters that the decision maker may learn in due course. When there is uncertainty, the statements below will be true "in expectation".

In general, assume that the value of getting  $x$  at time  $t$  as perceived at date  $s \leq t$  is

$$U(x, t|s) = \delta_{t,s}u(x)$$

where  $u$  is a known utility function and  $\delta_{t,s}$  is a known discount factor between the dates  $t$  and  $s$ , decreasing in  $t$  and increasing in  $s$ . Without loss of generality, take

$$\delta_{s,s} = 1.$$

A key question is how the decision maker at time  $s$  trades off the incomes at various future dates. In particular, it is important to know the equivalent of a dollar at  $t + 1$  at date  $t$

$$\frac{\delta_{t+1,s}}{\delta_{t,s}}.$$

I will refer to the ratio as impatience function; the decision maker at time  $s$  is indifferent between getting  $\frac{\delta_{t+1,s}}{\delta_{t,s}}$  at time  $t$  and getting a dollar at  $t + 1$ .

The key property of exponential discounting is that this impatience function does not depend on  $s$ . Indeed in the simplest case of constant  $r$ , that ratio is

$$\frac{\delta_{t+1,s}}{\delta_{t,s}} = e^{-r} \equiv \delta.$$

If  $r_t$  varies by time so that  $\delta_{t,s} = \exp(-\sum_s^t r_\tau)$ , then

$$\frac{\delta_{t+1,s}}{\delta_{t,s}} = e^{-r_t} \equiv \delta_t.$$

(In that case, we can also write  $\delta_{t,s} = \delta_s \cdots \delta_{t-1}$ .) This is the sense in which the exponential discounting exhibits *stationary impatience*:  $\delta_{t+1,s}/\delta_{t,s}$  is independent  $s$ .

Some experiments suggest that the subjects may exhibit *decreasing impatience*, in that  $\delta_{t+1,s}/\delta_{t,s}$  is decreasing in  $s$ . For example, a subject may prefer getting \$99 today instead of a \$100 tomorrow, while he prefers getting \$100 next Tuesday to getting \$99 next Monday (a day earlier). This does not contradict exponential discounting per se (as  $r$  may be smaller next Monday), but it would contradict exponential discounting if he changes his preferences when the next Monday comes and prefers \$99 on Monday to \$100 on Tuesday. That is,

$$\frac{\delta_{\text{Tuesday,Today}}}{\delta_{\text{Monday,Today}}} > \frac{99}{100} > \frac{\delta_{\text{Tuesday,Monday}}}{\delta_{\text{Monday,Monday}}}.$$

Such decreasing impatience is a property of what is called *hyperbolic discounting*:

$$\delta_{t,s} = (1 + a(t - s))^{-b/a}$$

where  $a$  and  $b$  are some positive parameters. Often one takes  $a = b$ , so that

$$\delta_{t,s} = 1 / (1 + a(t - s)).$$

A third functional form, called *quasi-hyperbolic discounting*, allows both decreasing and stationary discounting:

$$\delta_{t,s} = \begin{cases} 1 & \text{if } t = s \\ \beta\delta^{t-s} & \text{if } t > s \end{cases}$$

where  $\beta \in [0, 1]$  and  $\delta \in (0, 1)$ . The case  $\beta = 1$  corresponds to the exponential discounting, while the case  $\beta < 1$  corresponds to the decreasing impatience. (This model is also called  $\beta$ - $\delta$  model.) Here, the decreasing impatience is just a reflection of the *present bias*, in that the decision maker treats the present date  $s$  as special. I will use quasi-hyperbolic discounting to study the internal conflict in dynamic optimization problems next.

## 7.2 Optimal Consumption Plans

Consider a decision maker who wants to choose a consumption path for the rest of his life from time  $s$  on. As perceived from time  $s$ , her payoff from a consumption path

$$x = (x_0, x_1, \dots)$$

is

$$U(x|s) = \sum_{t=s}^{\infty} \delta_{t,s} u(x_t)$$

where  $u$  is a concave, increasing and continuously differentiable function. Imagine that he has initial wealth  $w_s$  that can be perfectly stored and that cannot bring any interest (for simplicity) so that he is constrained by

$$\sum_{t=s}^{\infty} x_t \leq w_s.$$

We want to study his optimal consumption paths under various specifications.

### 7.2.1 Optimal Consumption Path under Full Commitment

Now imagine that the decision maker can fully commit a consumption path. Then, his optimization problem is

$$\max_x U(x|s) \quad \text{subject to} \quad \sum_{t=0}^{\infty} x_t \leq w_s.$$

The optimal solution  $x^* = (x_s^*, \dots, x_t^*, \dots)$  is characterized by the first-order condition

$$\delta_{t,s} u'(x_t^*) = u'(x_s^*) \quad (\forall t \geq s) \quad (7.1)$$

and

$$\sum_{t=s}^{\infty} x_t^* = w_s.$$

Here, the first-order condition simply states that the marginal contribution  $\partial U(x|s)/\partial x_t$  of consumption at  $t$  is equal to the marginal contribution  $\partial U(x|s)/\partial x_s$  of the consumption at  $s$ . (This is because they enter the constraint  $\sum_{t=s}^{\infty} x_t \leq w_s$  as perfect substitutes. It would be slightly different otherwise.) Such equalization of marginal contributions is valid between all possible dates.

In particular, the first-order condition can be written as

$$\frac{u'(x_t^*)}{u'(x_{t+1}^*)} = \frac{\delta_{t+1,s}}{\delta_{t,s}} \quad (\forall t \geq s). \quad (7.2)$$

That is, the ratio of marginal contributions at consecutive dates is equal to the impatience for those dates. Hence, impatience plays a central role in optimal consumption plans.

I will next present two examples, both with CRRA.

**Example 7.1** *Take*

$$u(x) = \log(x).$$

*Then, the first order condition yields*

$$x_t^* = \delta_{t,s} x_s^* = \delta_{t,s} A w_s$$

where  $A = 1/(\sum_t \delta_{t,s})$ . Under the exponential discounting ( $\beta = 1$  in  $\beta$ - $\delta$  model), we have

$$x_t^{EXP,s} = \delta^{t-s} (1 - \delta) w_s.$$

Under quasi-hyperbolic discounting, we have  $A = 1/(1 + \beta\delta/(1 - \delta))$  and hence

$$x_t^{HD,s} = \begin{cases} \frac{1}{1 + \beta\delta/(1 - \delta)} w_s & \text{if } t = s \\ \frac{\beta\delta^{t-s}}{1 + \beta\delta/(1 - \delta)} w_s & \text{if } t > s \end{cases}.$$

**Example 7.2** *Now consider the more general class of CRRA functions:*

$$u(x) = x^{1-\rho}.$$

*Then, the first order condition yields*

$$x_t^* = \delta_{t,s}^\rho x_s^* = \delta_{t,s}^\rho A w_s$$

where  $A = 1/(\sum_t \delta_{t,s}^\rho)$ . That is, the solution is the same as the one under the logarithmic function—up to an augmentation of the discount factor to  $\delta_{t,s}^\rho$ . Under the exponential discounting ( $\beta = 1$  in  $\beta$ - $\delta$  model), the optimal consumption at  $t$  is

$$x_t^{EXP,s} = \delta^{(t-s)\rho} (1 - \delta^\rho) w_s.$$

Under quasi-hyperbolic discounting, we have  $A = 1/(1 + \beta^\rho \delta^\rho / (1 - \delta^\rho))$  and the optimal consumption at  $t$  is

$$x_t^{HD,s} = \delta_{t,s} = \begin{cases} \frac{1}{1 + \beta^\rho \delta^\rho / (1 - \delta^\rho)} w_s & \text{if } t = s \\ \frac{\beta^\rho \delta^{(t-s)\rho}}{1 + \beta^\rho \delta^\rho / (1 - \delta^\rho)} w_s & \text{if } t > s \end{cases}.$$

### 7.2.2 Internal Conflict and Time Inconsistency

I will now illustrate that when the time preferences do not exhibit stationary impatience, the preferences of decision maker at various dates will be different and be conflicting, in that they lead to contradicting decisions. Because of such a conflict, the preferences will also exhibit a form of dynamic inconsistency in that in the future the decision maker would like to change the contingency plans that he made previously, instead of executing the plan. In that case, one cannot simply use optimization techniques to find a solution. Instead, he would explicitly model the problem as a game between the different selves at various decision points, accounting for how much the decision maker can commit to his plans and what he knows about himself. This often requires a full-fledged game theoretical analysis—if not more.

I will first show that under stationary impatience the player does not exhibit internal conflict and we can use the solution to the optimization problem under full commitment without worrying about the commitment power of the decision maker. Assume that the time preferences exhibit stationary impatience as in the case of exponential discounting. Consider the selves at dates 0 and  $s$ . From the analysis above, the two selves have the same first-order condition for their optimization problems vis a vis two future dates  $t$  and  $t + 1$ :

$$\frac{u'(x_{t,s}^*)}{u'(x_{t+1,s}^*)} = \frac{\delta_{t+1,s}}{\delta_{t,s}} = \frac{\delta_{t+1,0}}{\delta_{t,0}} = \frac{u'(x_{t,0}^*)}{u'(x_{t+1,0}^*)},$$

where  $x_{t,s}^*$  denotes the optimal consumption at time  $t$  according to self  $s$ . Here, the middle equality is by stationary impatience. Under stationary impatience, the selves at 0 and  $s$  face the same trade off between the consumptions at future dates, as the above equality extends to any two future dates  $t$  and  $t'$ . They may choose different consumption paths only if the wealth at time  $s$  differs from the wealth the time 0 plan leaves for  $s$ . In particular, conditional on  $w_s$ , the selves at 0 and  $s$  would choose the same continuation paths after  $s$ . More generally, they have the same preference among the contingency plans after  $s$  (i.e. functions that map  $w_s$  to the consumption paths at  $s$  and thereafter). In that sense, the individual does not face any *internal conflict*. Moreover, his choices are *dynamically consistent*, in that the self at  $s$  simply executes the initial contingency plan made at 0 when the time  $s$  comes—instead of revising it.

**Example 7.3** *Imagine that the decision maker will retire at time  $s$  with some amount*

$w_s$  of money to consume for the rest of his life, where he may not necessarily know  $w_s$  before time  $s$ . Imagine that, having  $w_s$ , he can buy annuities at  $s$  that replace his money  $w_s$  with a stream  $(x_s, x_{s+1} \dots)$  of income such that  $\sum_{t=s}^{\infty} x_t \leq w_s$ . Suppose that he has logarithmic utility and exponential discounting—with  $\beta = 1$  in  $\beta$ - $\delta$  model. At time 0, he makes a contingency plan, deciding which annuity to buy when he retires as a function of  $w_s$ . As established previously, he chooses the annuity

$$x_t^{EXP,s} = \delta^{t-s} (1 - \delta) w_s, \quad t \geq s.$$

Now, when the date  $s$  comes and he retires with  $w_s$ , he has the same preference among the annuities (no internal conflict) and buys the annuity

$$x_t^{EXP,s} = \delta^{t-s} (1 - \delta) w_s, \quad t \geq s,$$

as he has planned originally (dynamic consistency).

**Example 7.4** Under quasi-hyperbolic discounting with  $\beta < 1$ , the things are quite different. Now, imagine that initially at time 0 he made a contingency plan (assuming he will stick to it when the retirement comes). As a function of  $w_s$ , he plans to buy the annuity

$$x_t^{EXP,s} = \delta^{t-s} (1 - \delta) w_s, \quad t \geq s.$$

This is because, at time 0, his preferences among the consumption paths starting date  $s$  coincide with a decision maker with exponential discounting  $\delta$  (make sure that you see this formally). When the retirement day  $s$  arrives, however, he would rather buy the annuity

$$x_t^{HD,s} = \begin{cases} \frac{1}{1+\beta\delta/(1-\delta)} w_s & \text{if } t = s \\ \frac{\beta\delta^{t-s}}{1+\beta\delta/(1-\delta)} w_s & \text{if } t > s \end{cases}, \quad t \geq s,$$

as established previously. His choice exhibits dynamic inconsistency, in that he revises his contingency plan instead of executing it. Moreover, he is conflicted: his preferences among the contingency plans are different, i.e., the two selves have different preferences among the annuities (for a fixed  $w_s$ ).

Such internal conflict is present whenever impatience is not stationary. Indeed, whenever the stationarity fails for some  $t$  and  $s$ , so that

$$\frac{\delta_{t+1,s}}{\delta_{t,s}} \neq \frac{\delta_{t+1,0}}{\delta_{t,0}},$$

the selves at 0 and  $s$  have different first-order conditions for their optimization problems under full commitment vis a vis the dates  $t$  and  $t + 1$ :

$$\frac{u'(x_{t,s}^*)}{u'(x_{t+1,s}^*)} = \frac{\delta_{t+1,s}}{\delta_{t,s}} \neq \frac{\delta_{t+1,0}}{\delta_{t,0}} = \frac{u'(x_{t,0}^*)}{u'(x_{t+1,0}^*)}.$$

They trade off the consumptions at  $t$  and  $t + 1$  differently, leading to a different solution.

Whenever such a conflict exists, it is not valid to use the solution to the optimization problem under full commitment. One then needs to treat different selves as different players (namely multi-self approach) and use an appropriate strategic solution concept that reflects the underlying assumptions, which of course vary from situation to situation. In the remainder, I will present some existing models that take such an approach.

### 7.2.3 Naive Solution under Quasi-hyperbolic Discounting

Consider the quasi-hyperbolic model with  $\beta < 1$ . Imagine that the decision maker naively thinks that he will commit to a consumption plan in the future although he will not have any commitment power in the future. Alternatively, imagine that at any moment he thinks that the present moment is special without taking into account that in the future he will always find the future present times special as well. Assume also CRRA utility, where  $\rho = 1$  corresponds to the logarithmic utility function.

As it has established already, at any time  $s$ , given the available wealth  $w_s$  left, the decision maker will plan on consuming

$$x_t^{HD,s} = \begin{cases} \frac{1}{1+\beta^\rho\delta^\rho/(1-\delta^\rho)}w_s & \text{if } t = s \\ \frac{\beta^\rho\delta^{(t-s)\rho}}{1+\beta^\rho\delta^\rho/(1-\delta^\rho)}w_s & \text{if } t > s \end{cases}$$

down the road, effectively consuming

$$x_s^{HD,s} = \frac{1}{1 + \beta^\rho\delta^\rho / (1 - \delta^\rho)}w_s$$

at any given  $s$ . At time 0, he plans to follow the consumption path

$$x_t^{HD,0} = \begin{cases} \frac{1}{1+\beta^\rho\delta^\rho/(1-\delta^\rho)}w_0 & \text{if } t = 0 \\ \frac{\beta^\rho\delta^{(t-s)\rho}}{1+\beta^\rho\delta^\rho/(1-\delta^\rho)}w_0 & \text{if } t > 0. \end{cases}$$

However, the actual consumption path (which he does not foresee) is

$$x_t^{NHD} = \frac{1}{1 + \beta^\rho\delta^\rho / (1 - \delta^\rho)} \left( \frac{\beta^\rho\delta^\rho / (1 - \delta^\rho)}{1 + \beta^\rho\delta^\rho / (1 - \delta^\rho)} \right)^t w_0, \quad t \geq 0.$$



To see this, observe that at each  $s$ , he consumes  $x_s^{HD,s}$  leaving  $w_s - x_s^{HD,s}$  to the next period. The wealth at  $t$  is

$$w_t^{NHD} = \left( \frac{\beta^\rho \delta^\rho / (1 - \delta^\rho)}{1 + \beta^\rho \delta^\rho / (1 - \delta^\rho)} \right)^t w_0.$$

By substituting  $w_t^{NHD}$  for  $w_t$  in the expression for  $x_t^{HD,t}$ , one obtains  $x_t^{NHD}$ .

Observe that, for large  $t$ , although time  $t$  self consumes more than originally planned as a fraction of his wealth at the time, his actual consumption is much less than planned because he is much poorer than time 0 self anticipates. This is because the selves in between consume more than he anticipates.

### 7.2.4 Sophisticated Solution

Now imagine that the decision maker with  $\beta < 1$  cannot commit to a consumption plan and he is aware of this fact. Since the selves now decide independently towards maximizing their own payoff, we model this situation as a game between the selves. That is, we model it as an extensive-form game with players  $s = 0, 1, 2, \dots$ . At date 0, time 0 self chooses a consumption level  $x_0$  with  $0 \leq x_0 \leq w_0$ , leaving

$$w_1 = w_0 - x_0$$

to the next date. At each date  $t > 0$ , the self  $s = t$  moves and chooses a consumption level  $x_t$  with  $0 \leq x_t \leq w_t$ , leaving

$$w_{t+1} = w_t - x_t$$

to the next date.

In order to capture the idea that the decision maker knows that he cannot commit and he will have a present bias in the future and that in the future too he will not be able to commit and will know that his future selves will have present bias, and so on, we use subgame-perfect Nash equilibrium as the solution concept. Any SPNE of the above game is called a *sophisticated solution*. If there is a finite deadline, there is a unique sophisticated solution and one can find the sophisticated solution using backward induction. In the current infinite-horizon game there can be multiple solutions, each yielding a different solution. Each solution tells a consistent scenario.

A special sophisticated solution is given by the stationary SPNE, in which the selves do not condition on the past behavior beyond the available wealth, which they take as the "state variable". Such an equilibrium can be taken as the limit of the backward induction solutions to the truncated games. Under a CRRA utility function, one can compute the stationary SPNE as follows. Assume that the equilibrium strategy of each self  $s$  is

$$x_s = \alpha w_s$$

for some constant  $\alpha$ , as a function of the available wealth  $w_s$ . Now, given this strategy for the future selves, if a time  $t$  self consumes  $x_t$ , then the future consumption and the wealth levels are as follows:

<u>date</u>	<u>wealth</u>	<u>consumption</u>
$t + 1$	$w_t - x_t$	$\alpha (w_t - x_t)$
$t + 2$	$(1 - \alpha) (w_t - x_t)$	$\alpha (1 - \alpha) (w_t - x_t)$
$t + 3$	$(1 - \alpha)^2 (w_t - x_t)$	$\alpha (1 - \alpha)^2 (w_t - x_t)$
...		
$t + s$	$(1 - \alpha)^{s-1} (w_t - x_t)$	$\alpha (1 - \alpha)^{s-1} (w_t - x_t)$

Hence, his payoff is

$$U(x_t|w_t) = u(x_t) + \sum_{s=1}^{\infty} \beta \delta^s u(\alpha (1 - \alpha)^{s-1} (w_t - x_t)).$$

The first order condition for the best response is

$$0 = U'(x_t|w_t) = u'(x_t) - \sum_{s=1}^{\infty} \beta \delta^s \alpha (1 - \alpha)^{s-1} u'(\alpha (1 - \alpha)^{s-1} (w_t - x_t)),$$

i.e.,

$$u'(x_t) = \sum_{s=1}^{\infty} \beta \delta^s \alpha (1 - \alpha)^{s-1} u'(\alpha (1 - \alpha)^{s-1} (w_t - x_t)). \quad (7.3)$$

After checking that this equation has indeed a solution in the form of  $x_t = a w_t$  for some constant  $a$  (that does not depend on  $w_t$ ), one finds the equilibrium  $\alpha$  by equation  $\alpha = a$ . In that case,  $\alpha$  is the solution to the equation

$$u'(\alpha w_t) = \sum_{s=1}^{\infty} \beta \delta^s \alpha (1 - \alpha)^{s-1} u'(\alpha (1 - \alpha)^s w_t).$$

Under CRRA, this equation simplifies to

$$\alpha^{-\rho} = \sum_{s=1}^{\infty} \beta \delta^s \alpha^{1-\rho} (1-\alpha)^{s(1-\rho)-1}.$$

Interestingly, under  $u(x) = \log(x)$ , we have

$$\begin{aligned} U(x_t|w_t) &= \log(x_t) + \sum_{s=1}^{\infty} \beta \delta^s \log(\alpha(1-\alpha)^{s-1}(w_t - x_t)) \\ &= \log(x_t) + \sum_{s=1}^{\infty} \beta \delta^s \log(w_t - x_t) + \sum_{s=1}^{\infty} \beta \delta^s \log(\alpha(1-\alpha)^{s-1}). \end{aligned}$$

Hence, the best response does not depend on  $\alpha$ . It is

$$x_t = \frac{1}{1 + \beta\delta/(1-\delta)} w_t,$$

as in the case of full commitment. Hence, the sophisticated solution that corresponds to the stationary SPNE is given by

$$\alpha = \frac{1}{1 + \beta\delta/(1-\delta)},$$

and the sophisticated solution coincides with the naive solution—although the rationales are quite different. For  $\rho < 1$ , the naive and the sophisticated solutions differ.

There can be many other sophisticated solutions. The following exercise proposes another solution.

**Exercise 7.1** *Beatrice has initial wealth of  $w_0$  and suffers from quasi-hyperbolic discounting. At any date  $s$ , her utility from a consumption stream  $x = (x_0, x_1, \dots)$  is*

$$U(x|s) = \ln(x_s) + \beta \sum_{k=1}^{\infty} \delta^k \ln(x_{s+k}),$$

where  $\beta, \delta \in (0, 1)$ . She gets return of  $r > 1$  from her savings so that her wealth at  $t+1$  is  $w_{t+1} = r(w_t - x_t)$  if her wealth at  $t$  is  $w_t$  and she consumes  $x_t$  at  $t$ .

1. Find her optimal consumption  $x^* = (x_0^*, x_1^*, \dots)$  for the case of exponential discounting (i.e.  $\beta = 1$ ).
2. Find a sophisticated-optimal consumption strategy for her in which the self at any given date  $s$  consumes  $\gamma w_s$ . Compute the constant  $\gamma$  and briefly verify that this is indeed a subgame-perfect equilibrium of the multi-agent game.

3. For  $\beta < 1$ , find conditions under which there is a sophisticated optimal consumption strategy under which she consumes according to  $x^*$  on the path of play. [Hint: Use the strategy in part (b) as the punishment for deviations from the desired path.]

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