Subjective Expected Utility

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We will go over Savage's subjective expected utility, and provide a very rough sketch of the argument he uses to prove his representation theorem. Aside from the lecture notes, good references are chapters 8 and 9 in "Kreps (1988): Notes on the Theory of Choice," and chapter 11 in "Gilboa (2009): Theory of Decision under Uncertainty."[1]

Let S be a set of states. We call events subsets of S, which we typically denote by A, B, C,... Write $\mathcal{S}$ for the collection of all events, that is, the collection of all subsets of S.[2] Let X a finite set of consequence.[3] A (Savage) act is a function $f : S \to X$, mapping states into consequences. Denote by F the set of all acts, and $\gtrsim$ is a preference relation on F. As usual, $\gtrsim$ represents the DM's preferences over alternatives. In Savage, alternative are acts.

Now we introduce an important operation among acts: For $f, g \in F$ and $A \in \mathcal{S}$ define the act $f_A g$ such that

$$f_A g(s) = \begin{cases} f(s) & \text{if } s \in A, \\ g(s) & \text{else.} \end{cases}$$

In words, the act $f_A g$ is equal to f on A, while equal to g on the complement on A.[4] This operation allows us to make "conditional" statements: if A is true, this happens; if not, this other thing happens.

Let's list Savage's axioms, which are commonly referred as P1, P2, ...

Axiom 1 (P1)**.** *The relation $\gtrsim$ is complete and transitive.*

Usual rationality assumption.

Axiom 2 (P2)**.** *For $f, g, h, h' \in F$ and $A \in \mathcal{S}$,*

$$f_A h \gtrsim g_A h \quad \Leftrightarrow \quad f_A h' \gtrsim g_A h'.$$

[1]Gilboa gives a broad overview, while Kreps provides more details and is more technical.
[2]Technicality: there are no algebras nor sigma-algebras in Savage's theory.
[3]Savage works with an arbitrary (possibly infinite) X. If so, another axiom, called P7, should be added to the list. It is a technical axiom, unavoidable but without essential meaning.
[4]Usually $f_A g$ is defined as the act which is equal to g on A, while equal to f otherwise. Of course the different in the definition is irrelevant.

"Sure-thing principle." To state the next axion, say that an event $A \in \mathcal{S}$ is **null** if $x_A y \sim y_A x$ for all $x, y \in X$.[5]

Axiom 3 (P3). *For $A \in \mathcal{S}$ not null event, $f \in F$ and $x, y \in X$,*

$$x \gtrsim y \quad \Leftrightarrow \quad x_A f \gtrsim y_A f.$$

Monotonicity (state-by-state) requirement.

Axiom 4 (P4). *For $A \in \mathcal{S}$ and $x, y, w, z \in X$ with $x \succ y$ and $w \succ z$*

$$x_A y \gtrsim x_B y \quad \Leftrightarrow \quad w_A z \gtrsim w_B z.$$

Provide a meaning to likelihood statement defined by betting behavior (see $\dot{\gtrsim}$ later).

Axiom 5 (P5). *There are $f, g \in F$ such that $f \succ g$.*

This is simply a non-triviality requirement.

Axiom 6 (P6). *For every $f, g, h \in F$ with $f \succ g$ there exists a finite partition $\{A_1, \ldots, A_n\}$ of S such that for all $i = 1, \ldots, n$*

$$h_{A_i} f \succ g \quad \text{and} \quad f \succ h_{A_i} g.$$

Innovative Savage's continuity axiom. From now on we will assume that $\gtrsim$ satisfies P1-P6. We will sketch Savage's argument to find a utility function $u : X \to \mathbb{R}$ and a probability $\mathbb{P} : \mathcal{S} \to [0, 1]$ such that for every $f, g \in F$

$$f \gtrsim g \quad \Leftrightarrow \quad E_{\mathbb{P}}[u(f)] \geqslant E_{\mathbb{P}}[u(g)].$$

The first part of the argument is devoted to elicit $\mathbb{P}$ (step 1 and 2). The second part, instead, find u by using the elicited $\mathbb{P}$ (step 3).

Step 1: Qualitative Probability

Take two consequences $x, y \in X$ such that $x \succ y$. Define the binary relation $\dot{\gtrsim}$ over $\mathcal{S}$ such that

$$A \dot{\gtrsim} B \quad \text{if} \quad x_A y \gtrsim x_B y.$$

From P4 the definition of $\dot{\gtrsim}$ does not depend on the choice of x and y. We interpret the statement "$A \dot{\gtrsim} B$" as "the DM considers event A at least as likely as event B." We do so because, according to $x_A y \gtrsim x_B y$, the DM prefers to bet on A rather than on B.

Claim 1. The relation $\dot{\gtrsim}$ satisfies the following properties:

[5]Null events will be the events with zero probability, events that the DM is certain they will not happen.

(i) $\dot{\succsim}$ is complete and transitive.

(ii) $A \dot{\succsim} \varnothing$ for all $A \in \mathcal{S}$.

(iii) $S \dot{\succ} \varnothing$

(iv) if $A \cap C = B \cap C = \varnothing$, then $A \dot{\succsim} B$ if and only if $A \cup C \dot{\succsim} A \cup B$.

(v) If $A \dot{\succ} B$, then there is a finite partition $\{C_1, \ldots, C_n\}$ of S such that

$$A \dot{\succ} B \cup C_k \quad \forall k = 1, \ldots, n.$$

This claim is relatively easy to prove. Because $\dot{\succsim}$ satisfies (i)-(iv), $\dot{\succsim}$ is called a **qualitative probability**. Savage's main innovation is (v), which comes from P6. Indeed, if only (i)-(iv) are satisfied, we may not be able to find a numerical representation of $\dot{\succsim}$.

Step 2: Quantitative Probability

A **quantitative probability** is a function $\mathbb{P} : \mathcal{S} \to [0,1]$ such that (i) $\mathbb{P}(S) = 1$, and (ii) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ when $A \cap B = \varnothing$.[6]

Claim 2. There exists a quantitative probability $\mathbb{P}$ representing the qualitative probability $\dot{\succsim}$:

$$A \dot{\succsim} B \quad \Leftrightarrow \quad \mathbb{P}(A) \geqslant \mathbb{P}(B) \quad \forall A, B \in \mathcal{S}.$$

Furthermore, for all $A \in \mathcal{S}$ and $\alpha \in [0,1]$ there exists $B \subset A$ such that $\mathbb{P}(B) = \alpha \mathbb{P}(A)$.

The second part of the claim says that $\mathbb{P}$ is **non-atomic**: any set with positive probability can be "chopped" to reduce its probability by an arbitrary amount. For instance, the uniform distribution has this property. Observe that there cannot be a non-atomic probability defined on a finite set (why?). Therefore, Savage's theory does not apply when S is finite. The proof of Claim 2 is somehow the core of Savage's argument, and the one thing should be remembered. Let's see an heuristic version of it:

"Proof". Fix an event B. We wish to assign a number $\mathbb{P}(B) \in [0,1]$ to B representing the likelihood of B according to DM. To do so, first we use (v) in Claim 1 to find for every $n = 1, 2, \ldots$ a partition $\{A_1^{(n)}, \ldots, A_{2^n}^{(n)}\}$ of S such that $A_1^{(n)} \dot{\sim} \ldots \dot{\sim} A_{2^n}^{(n)}$. Clearly we should assign probability $1/2^n$ to event $A_i^{(n)}$ for $i = 1, \ldots, 2^n$, and we can use this to assign a probability to B. Indeed, for every n we can find $k(n) \in \{1, \ldots, 2^n\}$ such that

$$\cup_{i=1}^{k(n)} A_i^{(n)} \dot{\succ} B \dot{\succsim} \cup_{i=1}^{k(n)-1} A_i^{(n)}.$$

[6] Technicality: note that P is additive, but possibly not sigma-additive.

This means that the probability of B should be at most $k(n)/2^n$ and at least $(k(n)-1)/2^n$. As n gets large, the bounds on the probability of B get closer and closer, so it makes sense to define

$$\mathbb{P}(B) = \lim_{n\to\infty} \frac{k(n)}{2^n}.$$

Then there is a substantial amount of work to verify that this guess for $\mathbb{P}(B)$ is actually correct, and the resulting $\mathbb{P}$ meets the requirements (additivity, representing $\dot{\succsim}$). $\square$

Step 3: Acts as Lotteries

Now that we have a probability $\mathbb{P}$ over S, it is "not hard" to elicit u. The idea is to find a way to apply the mixture space theorem. First we use acts to induce lotteries over X. For $f \in F$, define $P_f \in \Delta(X)$ as the distribution of f under P, that is: for all $x \in X$

$$P_f(x) = \mathbb{P}(\{s \in S : f(s) = x\}).$$

If the $\mathbb{P}$ we found is correct, better be the case that P_f and P_g contain all the information about f and g the DM uses to rank f and g. In fact:

Claim 3. For every $f, g \in F$, if $P_f = P_g$, then $f \sim g$.

This claim is very tedious to prove. It is easier to prove the following, using the fact that $\mathbb{P}$ is non-atomic (second part of Claim 2):

Claim 4. $\Delta(X) = \{P_f : f \in F\}$.

The claim says that for any lottery over X we can find an act generating it. Therefore, using Claim 3 and 4 we can well define a preference relation $\succsim^*$ over $\Delta(X)$ such that for $P, Q \in \Delta(X)$

$$P \succsim^* Q \quad \text{if there are } f, g \in F \text{ such that } P = P_f, Q = P_g \text{ and } f \succsim g.$$

Claim 5. The relation $\succsim^*$ on $\Delta(X)$ satisfies the assumption of the mixture space theorem (complete and transitive, continuity, independence).

Once we have Claim 5, we can apply the mixture space theorem and find $u : X \to \mathbb{R}$ such that for all $P, Q \in \Delta(X)$

$$P \succsim^* Q \quad \Leftrightarrow \quad \sum_{x\in X} P(x)u(x) \geqslant \sum_{x\in X} Q(x)u(x).$$

Now we have both $\mathbb{P}$ and u. Hence we can go back to $\succsim$ and verify that for all $f, g \in F$

$$f \succsim g \quad \Leftrightarrow \quad E_{\mathbb{P}}[u(f)] \geqslant E_{\mathbb{P}}[u(g)].$$

14.123 Microeconomic Theory III
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