1 Surplus Division

Output (surplus) to be divided among several agents.

- Issues: How to divide? How to produce? How to organize? Plus: adverse selection, moral hazard, ...
- Extremes easy: MAX s.t. Reservation Utility + Incentive Constraints
- Examples: Joint ownership, Unions, No ownership, Bargaining, Privatization, ..., essentially all the models we had considered without MAX objective; arbitration, laws, ...
- Objectives? Efficiency, Welfare, Fairness (!).
- Mechanisms to achieve: Bargaining, Arbitration, Auctions, Rationing, Lotteries, Markets, ...

2 Fair Distribution (Moulin'03)

2.1 Four Principles of Distributive Justice

1. Compensation
2. Reward
3. Exogenous rights
4. Fitness

Pluto's Flute: 4 children

1. (poor) no toys
2. (worked hard) cleaned and fixed it
3. (owner) father's flute
4. (efficient user) can play

- Compensation: Ex post equality

Goal: To equalize distribution of a higher-order characteristic

Justifies: Disproportional use of resources

Examples: Different shares of food for infants, pregnant women, adult males; More medical attention to ill; More attention to Handicapped; affirmative action for socioeconomically disadvantaged.

Justification for macroeconomic redistributive policies (other j..?): tax breaks, welfare support, medical aid.

For each i, $v_i = u_i(y_i)$. Choose $\{y_i\}$ to equate v_i's.

Egalitarian objective.

Mechanisms: handicaps, unequal shares, subsidies to certain groups,...

- Reward

Unequal treatment is (morally) relevant as a reward (punishment) for actions (behavior).

Past sacrifices by soldiers lead to preferential treatment today; prizes for achievements; higher insurance premium for reckless drivers; no organ transplants for criminals.

Difficult issue is when the outcome is a product of efforts of multiple agents. How to compensate? (public goods, tragedy of commons)

Bargaining outcomes, Shapley value, Exogenous (in expectation) budget breakers,...

Another issue: Outside opportunities.

- Exogenous rights (equality ex ante?)

Property rights (and liabilities)

Fairness: freedom of speech, religion, access to education (unrelated to IQ), voting rights (unrelated to characteristics, both external (rich, male, educated) and internal (smart, caring, voting)), equal duties, political representation, one share–one vote.

Unfairness: order of Priority based on: Seniority, Social standing (cast structure), size of representation,...

- Fitness (Efficiency)

Who is the best, who values it the most, who in the most need; Child to the true mother, ... a drink to a drunk, ...

Distinguish: sum-fitness (MAX SUM) vs efficiency-fitness (Pareto Optimality)

Utilitarian Objective.

Flute example: ?

- Examples

Lifeboat (sinking ship):

Women and children, Old people, Crew, strong men, “generals”: who first?

Food rationing in besieged town;

Limited medical resources:

Immigration, college admissions, tickets to shows...

- Queuing and auctioning

How these perform on different criteria? (think overbooked plane)

- Political rights (voting)

Plato: philosophers should rein

- Joint venture (Excess)

Teresa (piano) earns \$50K alone;

David (violin) earns \$100K alone;

Together: \$210K. Split?

1. Proportional solution

Stand-alone salaries are proxies for individual contribution

$$y_i = \frac{x_i}{\sum x_i} T$$

2. Status quo ante solution

$$y_i = x_i + \frac{1}{n}\left(T - \sum x_j\right)$$

3. Equal division (egalitarian) modified: uniform gains solution

$$\begin{aligned} y_i &= \max\{\lambda, x_i\} \\ \sum \max\{\lambda, x_i\} &= T. \end{aligned}$$

- Joint venture (Deficit)

E.g. Bankruptcy

Equal division

Proportional Division

Uniform Losses

...Lotteries

3 The Shapley Value

The problem of the Commons (a joint production process)

Sharing of joint costs or benefits.

What is a fair assessment of individual responsibilities or contributions.

Extreme: "Without me you are nothing"

- Joint venture (revisited)

T and D share an office, need good connection.

T (D) needs a link that costs $c_T < c_D$. (stand-alone costs). There is a single cable outlet.

Additional cost $\delta > 0$ to connect both, $C = c_T + c_D + \delta$.

Which solution to use?

Comparative statics (suppose the company drives c_T to 0).

Proportional division: $P_T = 0$, $P_D = c_D + \delta$ (full externality). Surely, T has to pay some.

Uniform gains: $P_T = \delta$, $P_D = c_2$ if $\delta < c_2$ (equal otherwise).

Equal surplus: (sensible) $P_i = c_i + \delta/2$.

(Status-quo plus Nash B ?)

3.1 The Shapley Value: Definition

Cost interpretation: each agent wants one unit of service (equal ex ante ownership)

$N = \{1, 2, ..., n\}$,

coalition is a subset $S \subseteq N$.

For each S, there is $C(S)$—stand-alone cost of serving S.

(characteristic function in general)

Solution is *Expected marginal cost.*

Let A_i be the set of coalitions NOT containing i; $A_i(m)$ is the set of coalitions from A_i of size m.

Shapley Value is

$$x_i = \sum_{m=0}^{n} \sum_{S \subseteq A_i(m)} \frac{m!(n-m-1)!}{n!} \left[C(S \cup \{i\}) - C(S) \right].$$

The coefficient comes from an arbitrary order of players in S (those who joined S before i), and of players not in $S \cup \{i\}$ (those who join N later).

It is presumed that the grand coalition will form. (alternative) definition of Shapley value is the average over all possible orders of players of the marginal impact of a given player.

- Example: Runway construction

Airline A needs short runway only, B medium, Z long.

$C(A) = 1000$, $C(B) = C(AB) = 3000$,

$C(Z) = C(ABZ) = C(AZ) = C(BZ) = 6000$.

How to divide? 6 possible random orders:

A : Only when first has marginal cost:

$$x_A = \frac{1}{6}(1000_{ABZ} + 1000_{AZB}) = \frac{1000}{3};$$

B : Only when first or second (after A) has added cost:

$$x_B = \frac{1}{6}(2 \times 3000_{B.} + 2000_{ABZ}) = \frac{1000}{3} + \frac{2000}{2};$$

C :

$$\begin{aligned} x_Z &= \frac{1}{6}\begin{pmatrix} 2 \times 6000_{Z.} + 5000_{AZB} \\ +3000_{BZA} + 2 \times 3000_{..Z} \end{pmatrix} \\ &= \frac{1000}{3} + \frac{2000}{2} + 3000. \end{aligned}$$

3.2 Stand-alone property of Shapley Value

Subadditivity: $C(S \sqcup T) \leq C(S) + C(T)$

Then, $C(N) \leq \sum C(i)$;

Superadditivity: $C(S \sqcup T) \geq C(S) + C(T)$

Then, $C(N) \geq \sum C(i)$.

Stand-alone test: C subadditive $\Rightarrow y_i \leq C(i)$; C superadditive $\Rightarrow y_i \geq C(i)$.

- Shapley Value meets S-A test.

Lots of other properties (see axioms next)

3.3 Shapley Value: Axiomatic Approach

Variety of ways, Original axioms are Equal treatment of equals, Dummy, and Additivity.

Generic solution, $\{\gamma_i\}_{i=1}^n$, $\sum \gamma_i = C(N)$.

Equal treatment of equals: if i, j are equal (exch) w.r. (C, N), then $\gamma_i = \gamma_j$.

Dummy: (only axiom that contains reward principle): suppose i is such that for all $S \subseteq A_i$, $C(S \cup \{i\}) - C(S) = 0$, then $\gamma_i = 0$.

Additivity: (structural invariance) $\gamma(C^1 + C^2, N) = \gamma(C^1, N) + \gamma(C^2, N)$.

THM: Shapley Value is the only solution satisfying ABOVE AXs.

Marginalism (in place of D and A): If for C^1 and C^2 marginal impacts of player i are the same, then $\gamma_i(C^1) = \gamma_i(C^2)$.

Equal Impact (variable population axiom; fairness) Impact of removing j on i's share is the same as impact of removing i on j's share.

Potential: exists a real-valued potential function defined for all (N, C)

$\gamma_i(N, C) = P(N, C) - P(N \setminus i, C^{-i})$ for all N, i, C.

Extensions: Unequal exogenous rights.

Recently: Maskin extension to externalities. Also, different interpretation, an aggregate market-value added (the lowest value (highest cost)) that agent accepts to join. (predetermined random order of sequential offers).

4 Nash Bargaining solution

U is utility possibility set; u^0 is a status quo.

Bargaining solution is a rule f that assigns a solution vector $f(U, u^0) \in U$ to every bargaining problem (U, u^0).

Variants: Egalitarian, Utilitarian, Nash, ...

Properties (axioms):

Independence of utility origins: Consider $U' = U + \boldsymbol{\alpha}$ ($u_i' = u_i + \alpha_i$), then $f_i(U', u^0 + \boldsymbol{\alpha}) = f_i(U, u^0) + \alpha_i$ for all i.

(the bargaining solution does not depend on absolute scales of utility)

Normalize treat-point: $u^0 = 0$.

Independence of utility units: $U' = \boldsymbol{\beta} U$ with $\boldsymbol{\beta} > \mathbf{0}$, then $f_i(U') = \beta_i f_i(U)$ for all i. (no personal comparisons of utilities)

Pareto property (weak): $f(U)$ is such that there is no $u^* \in U$, $u_i^* > f_i(U)$ for all i.

Symmetry: ..

Individual rationality: $f(U) \geq 0$.

Independence of irrelevant alternatives: If $U' \subset U$, and $f(U) \in U'$, then $f(U') = f(U)$.

- Egalitarian solution:

Satisfies **IIA**, does not satisfy **IIU**.

- Utilitarian solution: ?

- Nash solution:

$$\max u_1 u_2 ... u_n$$

THM: The Nash solution is the only bargaining solution satisfying the above.,

- Contrast vs Walrasian equilibrium (competitive markets in general)

Generically different from Walrasian Equilibrium (with few participants)

Are assumptions of "fair" or "competitive" prices appropriate?