14.13 Lecture 9

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What about non-Gumbel noise?

- Definition. A distribution is in the domain of attraction of the Gumbel if and only if there exists constants A_n, B_n such that for any x

$$\lim_{n \to \infty} P\left(\max_{i=1,\ldots,n} \varepsilon_i \leq A_n + B_n x\right) = e^{-e^{-x}}.$$

  when ε_i are iid draws from the given distribution.

- Fact 1. The following distributions are in the domain of attraction of a Gamble: Gaussian, exponential, Gumbel, lognormal, Weibull.

- Fact 2. Bounded distributions are not in this domain.

- Fact 3 Power law distributions ($P(\epsilon > x) \sim x^{-\zeta}$ for some $\zeta > 0$) are not in this domain.

- Lemma 1. For distributions in the domain of attraction of the Gumbel $F(x) = P(\varepsilon < x)$ take $\bar{F}(x) = 1 - F(x) = P(\varepsilon \geq x)$, and $f = F'$. Then A_n, B_n are given by

$$\bar{F}(A_n) = \frac{1}{n}$$

$$B_n = \frac{1}{nf(A_n)}$$

indeed $E[\overline{F}(M_n)] = \frac{1}{n+1}$. In general, order $\epsilon_{1;n} \geq \epsilon_{2;n} \geq ... \geq \epsilon_{n;n}$, then $F(\epsilon_{k;n}) \simeq 1 - \frac{k}{n}$

- Lemma 2

$$\lim_{n\to\infty} P\left(\max_{i=1,\dots n} \varepsilon_i + q_i \leq A_n + B_n y + q_n^*\right) = e^{-e^{-y}}$$

with

$$e^{q_n^*/B_n} = \frac{1}{n}\sum e^{q_i/B_n}$$

- Proposition.

$$D_1 = P\left(q_1 - p_1 + \sigma\varepsilon_1 > \max_{i=2,\ldots,n} q_i - p_i + \sigma\varepsilon_i\right)$$

For $n \to \infty$, $\lim D_1/\bar{D}_1 = 1$ where

$$\bar{D}_1 = \frac{e^{\frac{q_1 - p_1}{B_n \sigma}}}{\sum_{i=1}^{n} e^{\frac{q_i - p_i}{B_n \sigma}}} \simeq D_1.$$

- Example 1. Exponential distribution $f(x) = e^{-(x+1)}$ for $x > -1$ and equals 0 for $x \leq -1$. then, for $x > -1$

$$\bar{F}(x) = P(\varepsilon > x) = \int_x^\infty e^{-(x+1)} dy$$
$$= \left[-e^{-(x+1)}\right]_x^\infty = e^{-(x+1)} = f(x).$$

Thus

$$\bar{F}(A_n) = \frac{1}{n},$$

and

$$A_n = -1 + \ln n$$

and

$$B_n = \frac{1}{nf(A_n)} = 1$$

- Example 2. Gaussian. $f(x) =$, $\bar{F}(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds$. For large x, the cumulative $\bar{F}(x) \sim \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi} x}$. Result

$$A_n \sim \sqrt{2 \ln n}$$
$$B_n \sim \frac{1}{\sqrt{2 \ln n}}$$

Optimal prices satisfy

$$\max_i \frac{(p_i - c_i)\, e^{\frac{q_i - p_i}{B_n \sigma}}}{\sum e^{\frac{q_j - p_j}{B_n \sigma}}} = \max\, (p_i - c_i)\, \bar{D}_1 = \pi_i$$

- Same as for Gumbel with $\sigma' = B_n \sigma$.

- Thus

$$p_i - c_i = B_n \sigma$$

– Gumbel

$$p_i - c_i = \sigma$$

– Exponential noise

$$p_i - c_i = \sigma$$

– Gaussian

$$p_i - c_i = \frac{1}{\sqrt{2 \ln n}} \sigma$$

and competition almost does not decrease markup (beyond markup when there are already some 20 firms).

– note that in the Cournot competition

$$p_i - c_i \sim \frac{1}{n}$$

- Example. Mutual funds market.
  - Around 10,000 funds. Fidelity alone has 600 funds.
  - Lots of fairly high fees. Entry fee 1-2%, every year management fee of 1-2% and if you quit exit fee of 1-2%. On the top of that the manager pays various fees to various brokers, that is passed on to consumers.
  - The puzzle – how all those markups are possible with so many funds?
  - Part of the reason for that many funds is that Fidelity and others have incubator funds. With large probability some of them will beat the market ten years in a row, and then they can propose them to unsophisticated consumers.