Economics of Networks
Network Formation

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Agenda

- Strategic network formation
- Pairwise stability
- One-sided link formation
- A game played on an endogenous network

Reading: Jackson, chapters 6 and 11

Motivation

First half of course covered several models for generating random networks, capture observed features

Why do networks form? What determines their structure?

- Coauthor networks
- Firm R&D networks
- Financial networks

Introduce ideas from cooperative game theory to study strategic choices about link formation

A Network Formation Game

Set of players N, set of possible graphs $\mathcal{G}$ (undirected)

Payoff for player i:

$$u_i : \mathcal{G} \to \mathbb{R}$$

Utility for every possible network

Will consider different specific payoff functions, different protocols for forming links

Pairwise Stability

Problem with Nash equilibrium in this setting, will apply a cooperative solution concept: pairwise stability

Suppose players simultaneously announce with whom they want to form links

- Huge multiplicity if link formation requires both parties to announce

Want to account for strategic incentives, but allow for better coordination

Pairwise stability: a network is stable if no one wants to delete a link and no pair wants to form a link

Pairwise Stability

Definition

A network G is pairwise stable if:

- For all $ij \in G$, we have $u_i(G) \geq u_i(G - ij)$, and
- For all $ij \notin G$, if $u_i(G + ij) > u_i(G)$, then $u_j(G + ij) < u_j(G)$

If a link is present, neither player benefits from its removal

If a link is not present and one agent benefits from adding it, the other is worse off from adding it

Unilateral deletion, need agreement for addition

Efficiency

Are stable networks efficient?

Definition

A network G is efficient if

$$\sum_{i=1}^{n} u_i(G) \geq \sum_{i=1}^{n} u_i(G') \quad \forall\, G' \in \mathcal{G}$$

A network G is Pareto efficient if there is no $G' \in \mathcal{G}$ such that $u_i(G') \geq u_i(G)$ for all i, with strict inequality for at least one i.

Efficiency implies Pareto efficiency

- Is the reverse true?

Example: Distance-Based Utility

Suppose we have

$$u_i(G) = \sum_{j \neq i} b(l_{ij}(G)) - d_i(G)c$$

- $c > 0$ is the cost of making a connection
- $d_i(G)$ is the number of neighbors i has in G
- $l_{ij}(G)$ is the distance between i and j in G (take $l_ij(G) = \infty$ if i and j are not connected)
- $b : \mathbb{N} \to \mathbb{R}$ is a benefit function depending on distance, assume strictly decreasing and $b(\infty) = 0$

An efficient network maximizes

$$U = \sum_{i=1}^{n} \left[\sum_{j \neq i} b(l_{ij}(G)) - d_i(G)c \right]$$

Pairwise Stability and Efficiency

Suppose that

$$b(1) < c < b(1) + (n-2)b(2)$$

Efficient network is a star network

- Peripheral players i earn

$$u_i(G) = b(1) - c + (N-2)b(2)$$

- Central player earns $(n-1)(b(1)-c)$

Total utility

$$U = (n-1)\left[2b(1) - 2c + (n-2)b(2)\right]$$

Pairwise Stability and Efficiency

Given symmetry, sufficient to check that we cannot increase U by adding or subtracting one link

Suppose we remove a link, new utlity

$$U' = (n-2)\left[2b(1) - 2c + (n-3)b(2)\right]$$

We have

$$\begin{aligned} U' - U &= -2b(1) + 2c - 2(n-2)b(2) \\ &= -2\left(b(1) - c + (n-2)b(2)\right) < 0 \end{aligned}$$

Since $c < b(1) + (n-2)b(2)$

Pairwise Stability and Efficiency

If we add a link, we reduce the distance between just two of the players

New utility

$$U'' = (n-3)\left[b(1) - c + (n-2)b(2)\right] + (n-1)\left[b(1) - c\right] \\ + 2\left[2b(1) - 2c + (n-3)b(2)\right]$$

Difference

$$U'' - U = 2\left(b(1) - c\right) - 2b(2) < 0$$

since $b(1) < c$

Pairwise Stability and Efficiency

However, the star network is *not* pairwise stable

Central player earns utility

$$u^*(G) = (n-1)\left(b(1) - c\right) < 0$$

In general, stable networks $\neq$ efficient networks

Intuitively, links to the central player create positive externalities for others, reduces distances for all

- Fails to internalize these benefits

Pairwise Stability and Efficiency

Theorem

In the distance-based utility mode, the unique efficient network is

- *The complete network if* $b(2) < b(1) - c$
- *A star on all nodes if* $b(1) - b(2) < c < b(1) + (n-2)b(2)$
- *The empty network if* $c > b(1) + (n-2)b(2)$

In contrast, stable networks have the following properties

- *There is at most one non-empty component*
- *If* $b(2) < b(1) - c$*, the complete network is uniquely stable*
- *If* $b(1) - b(2) < c < b(1)$*, the star network is stable*
- *If* $b(1) < c$*, every node has either no links or at least two links*

Pairwise stable networks are inefficient exactly when

$$b(1) < c < b(1) + (n-2)b(2)$$

Pairwise Stability and Efficiency

Positive externalities $\implies$ too few links in equilibrum

- What about negative externalities?

Look at the "coauthor model" alert(Jackson and Wolinsky, 1996)

- If your coauthor has lots of other coauthors, less time to spend on your project

Payoff for player i

$$u_i(G) = \sum_{j\,:\,ij \in G} \left(\frac{1}{d_i(G)} + \frac{1}{d_j(G)} + \frac{1}{d_i(G)d_j(G)} \right)$$

for $d_i(G) > 0$. Set $u_i(G) = 1$ if $d_i(G) = 0$

Pairwise Stability and Efficiency

Theorem (Jackson and Wolinsky)

In the coauthor model, if n is even, the efficient network structure consists of $\frac{n}{2}$ distinct pairs. If a network is pairwise stable and $n \geq 4$, the network is inefficient and can be partitioned into cliques, each with a different number of members.

People have too many coauthors

- Would be efficient to work with one partner, but players form larger groups

Proof

To maximize efficiency, we maximize

$$\sum_{i=1}^{n} u_i(G) = \sum_{i\,:\,d_i(G)>0} \sum_{j\,:\,ij\in G} \left(\frac{1}{d_i(G)} + \frac{1}{d_j(G)} + \frac{1}{d_i(G)d_j(G)} \right)$$

Sum over first two terms bounded by $2n$

Sum over last term bounded by n, equality iff $d_i(G) = d_j(G) = 1$ for all ij

- Only happens if every play has one neighbor

Proof

Now pairwise stable networks. Suppose $ij \notin G$. Player i wants to create link ij iff

$$u_i(G + ij) > u_i(G)$$

which is true if

$$\sum_{k \neq j,\, ik \in G} \left(\frac{1}{d_i(G) + 1} + \frac{1}{d_k(G)} + \frac{1}{(d_i(G) + 1) d_k(G)} \right)$$
$$+ \frac{1}{d_i(G) + 1} + \frac{1}{d_j(G) + 1} + \frac{1}{(d_i(G) + 1)(d_j(G) + 1)}$$
$$> \sum_{k \neq j,\, ik \in G} \left(\frac{1}{d_i(G)} + \frac{1}{d_k(G)} + \frac{1}{d_i(G) d_k(G)} \right)$$

Proof

Simplifying gives

$$\frac{1}{d_j+1}\left(1+\frac{1}{d_i+1}\right) > \left(\frac{1}{d_i}-\frac{1}{d_i+1}\right)\left(\sum_{k\neq j,\, ik\in G}\frac{1}{d_k}\right)$$

Multiply by $d_i(G)+1$ to obtain

$$\frac{d_i(G)+2}{d_j(G)+1} > \frac{1}{d_i(G)}\left(\sum_{k\neq j,\, ik\in G}\frac{1}{d_k(G)}\right)$$

(if $d_i(G)=0$, take RHS $=0$)

First show if $d_i(G)=d_j(G)$, then i and j want to form a link

- If $d_i(G)=d_j(G)$, LHS >1 for both, RHS <1 for both

Proof

To complete the argument, show if i and j share a neighbor h with $d_h \geq \max\{d_i, d_j\}$, then i and j want to connect

- Implies connected components are cliques

If $d_i \geq d_h - 1$, then $\frac{d_i+2}{d_j+1} \geq 1$

- If inequality is strict, we are done
- If not, then $d_j \geq 2$ and $d_h \geq 2$, implying

$$\frac{1}{d_i(G)} \left(\sum_{k \neq j,\, ik \in G} \frac{1}{d_k(G)} \right) < 1$$

  and we are done

If $d_i \geq d_h - 1$, then i wants to link to j; proof for $d_i < d_h - 1$ left as an exercise

Existence of Pairwise Stable Networks

Stable networks exist in our examples, is this always the case?

No. Suppose there are 4 players and:

- Forming a link costs 5 to each player
- Utility from being isolated is 0
- Utility from being linked in an isolated pair is 12
- Utility from being connected (diretly or indirectly) to two others is 16
- Utility from being connected to all three others is 18

There is no stable network

Potential Functions

How to guarantee existence?

- Think back to potential games

Definition

Networks G and G' are *adjacent* if either $G' = G + ij$ or $G = G' + ij$ for some ij. We say that G' adjacent to G *defeats* G if either

- $G' = G - ij$ and $u_i(G') > u_i$
- $G' = G + ij$ and both $u_i(G') \geq u_i(G)$ and $u_j(G') \geq u_j(G)$, with at least one strict inequality

A network is pairwise stable iff there is no adjacent network that defeats it

Potential Functions

A sequence of adjacent networks $(G_1, G_2, ..., G_K)$ is an *improving path* if G_{k+1} defeats G_k

If no pairwise stable network exists, there must be an improving cycle: an improving path with $G_1 = G_K$

Can rule this out if utilities come from a potential function: $\phi : \mathcal{G} \rightarrow \mathbb{R}$ is an ordinal potential if G' defeats G if and only if $\phi(G') > \phi(G)$

Proposition

If the network formation game has an ordinal potential, there are no improving cycles.

Directed Networks and Nash Stability

In some applications, links are made unilaterally

- Paper citation
- Webpage linking
- Following on social media

A different game: let $\mathcal{G}$ be the set of directed networks on the set of players N

- Players simultaneously propose sets of directed links
- If i chooses to form link ij, it gets formed

Unilateral formation, look at equilibria

Directed Networks and Nash Stability

Set of pure strategies for player i is $S_i = 2^{N \setminus \{i\}}$

- Link ij forms if $j \in s_i$
- Network is $G(s) = \{ij \, : \, j \in s_i \text{ for some } i\}$

Definition

A directed network G is (strictly) *Nash stable* if $u_i(G)(>) \geq u_i(G')$ for each i and all G' that differ from G only on links originating from i.

Directed Networks and Nash Stability

A simple example: write $R_i(G)$ for the number of players reachable from i through directed paths

- Assume payoff to player i is

$$u_i(G) = R_i(G) - cd_i(G)$$

Theorem (Bala and Goyal, 2000)

The unique efficient network structure is an n-player wheel if $c < n-1$ and an empty network if $c > n-1$. Moreover,

- *If $c < 1$, then n-player wheels are the only strictly Nash stable networks*
- *If $1 < c < n-1$, then n-player wheels and empty networks are strictly Nash stable*
- *If $c > n-1$, then the empty network is uniquely Nash stable*

Network Formation and Network Effects

As discussed in the lectures on network effects, network structure can affect our incentives to adopt products or engage in certain behaviors (e.g. crime)

Typically the network is endogenous

- Choice to form links influenced by anticipated outcome of strategic interactions

A simple framework to think about this:

- Unit mass of players
- Player i invests productive effort $k_i \in \mathbb{R}^+$ and social effort $s_i \in \mathbb{R}^+$
- Write simply k and s for the average productive and social efforts of other players

Network Formation and Network Effects

Player i earns utility

$$u_i(s_i, k_i, s, k) = k_i - \frac{1}{2}ck_i^2 + \alpha s_i\sqrt{k_i k} - \frac{1}{2}s_i^2$$

- Private benefit k_i
- Cost of productive effort $\frac{1}{2}ck_i^2$
- Complementarities scale with social effort
- Cost of social effort $\frac{1}{2}s_i^2$

Players make choices simultaneously, study Nash equilibria

Network Formation and Network Effects

Decision problem is symmetric, have $s_i = s$ and $k_i = k$ in equilibrium

Best response implies

$$0 = 1 - ck_i + \alpha\frac{s_i\sqrt{k}}{2\sqrt{k_i}}, \qquad 0 = \alpha\sqrt{k_i k} - s_i$$

Taking $k_i = k$ and $s_i = s$ gives

$$0 = 1 - ck + \alpha\frac{s}{2}, \qquad 0 = \alpha k - s$$

In equilibrium

$$k = \frac{2}{2c - \alpha^2}, \qquad s = \frac{2\alpha}{2c - \alpha^2}$$

Equilibrium Welfare

Equilibrium payoffs:

$$u = (1+\alpha s)k - \frac{1}{2}ck^2 - \frac{1}{2}s^2$$
$$= \frac{2c+\alpha^2}{2c-\alpha^2}\frac{2}{2c-\alpha^2} - \frac{2c+2\alpha^2}{(2c-\alpha^2)^2}$$
$$= \frac{2c}{(2c-\alpha^2)^2}$$

Naturally decreasing in c, increasing in α (need $\alpha < \sqrt{2c}$ for this to make sense)

Ratio $\frac{s}{k}$ increasing in α

Efficiency

What are the efficient effort levels?

Maximize

$$u = k - \frac{1}{2}ck^2 + \alpha sk - \frac{1}{2}s^2$$

First order conditions

$$0 = 1 + \alpha s - ck, \qquad 0 = \alpha k - s$$

Solving yields

$$k = \frac{1}{c - \alpha^2}, \qquad s = \frac{\alpha}{c - \alpha^2}$$

Underinvestment in equilibrium due to positive externality

Network Formation and Network Effects

Generalizations:

- Heterogeneous types of players
- Ability to discriminate in linking effort

An important application: academic peer effects

- Peer effects in the classroom
- Peer effects from parental investments

Carrell et al. (2013), U.S. Airforce study